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ON THE CENTRAL LIMIT PROBLEM FOR PROCESSES OF ZERO ENTROPY

Dalibor VOLNÝ

Abstract: In this paper we show that a strictly stationary sequence of random variables with zero entropy can belong to the domain of partial attraction of a uniform distribution. The dynamical system which is used in the construction is a rotation.

Key words and phrases: Central limit problem, strictly stationary process of zero entropy, dynamical system.

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Let $(\Omega, \mathcal{A}, T, \mu)$ be a dynamical system where $(\Omega, \mathcal{A}, \mu)$ is a probability space ($\mathcal{A}$ is a σ-algebra of subsets of Ω and μ is a probability measure) and T is a one-to-one bimeasurable and measure preserving transformation of Ω onto Ω.

For $f \in L^2(\mu)$, the sequence $(f \circ T^i; i \in \mathbb{Z})$ is strictly stationary. It is proved in [1] that there exists an invariant σ-algebra $\mathcal{M} \subset \mathcal{A}$ (i.e. $\mathcal{M} \subset T^{-1}\mathcal{M}$) such that f is measurable with respect to the σ-algebra $\sigma \bigcup_{i \in \mathbb{Z}} T^i \mathcal{M}$ and the function $f_{-\infty} = E(f \mid \bigcap_{i \in \mathbb{Z}} T^i \mathcal{M})$ is measurable with respect to the Pinsker σ-algebra. For $f_i = E(f \mid T^{-i-1}\mathcal{M}) - E(f \mid T^{-i}\mathcal{M})$, $i \in \mathbb{Z}$, it holds $f = f_{-\infty} + \sum_{i \in \mathbb{Z}} f_i$ mod μ. In accordance with [1] we say that $f_{-\infty}$ is the absolutely undecomposable and $\sum_{i \in \mathbb{Z}} f_i$ is the difference decomposable part of f. According to [1], the decomposition of f into a sum of an absolutely undecomposable and a difference decomposable part always exists and is unique with respect to the equality mod μ (note that for each f_i, $i \in \mathbb{Z}$, the functions $f_i \circ T^j$, $j \in \mathbb{Z}$, form a martingale difference sequence).

Many interesting results have appeared when investigating the central limit problem for strictly stationary sequences of random variables (a review of this research can be found e.g. in the fifth chapter of [2]). According to [1], the achieved results concern the case of functions with degenerate difference decomposable parts (in the sense that their standardized sums converge weakly to zero). The aim of this paper is to give examples of functions f which are measurable with respect to the Pinsker σ-algebra (if they are from $L^2(\mu)$ they are absolutely undecomposable) for which the sequences $\mu(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} f \circ T^j)^{-1}$ have nondegenerate limit points.

Let $\Omega = \langle -1,1)$, $\mathcal{B}$ be a σ-algebra of Borel sets on Ω and $\mu = \frac{1}{2}m$ be a probability measure on $(\Omega, \mathcal{B})$ where m is the Lebesgue measure.

We define a function ψ on the interval $\langle -1,3)$ such that $\psi(\omega) = \omega$ for $\omega \in \langle -1,1)$ and $\psi(\omega) = \omega - 2$ for $\omega \in \langle 1,3)$. If $0 \le a < 2$ we define $T_a(\omega) = \psi(\omega + a)$. Evidently, T_a is a one-to-one bimeasurable and measure preserving transformation of Ω onto itself. According to [3], the dynamical system $(\Omega, \mathcal{B}, T_a, \mu)$ is of zero entropy and it is ergodic for a irrational.

For n = 1,2,... and $\omega \in \Omega$ we define $r_n(\omega) = \omega - \frac{[n\omega]}{n}$ (where $[n\omega]$ is the integer part of the number $n\omega$). Evidently, for $\omega \in \Omega$ it is $0 \le r_n(\omega) < \frac{1}{n}$ and there is a unique number $j \in \{-n,\dots,n-1\}$ such that $\omega = \frac{j}{n} + r_n(\omega)$. One can easily see that whenever $a = \frac{k}{n}$, $k \in \{1,\dots,2n-1\}$, we have $r_n(T_a^i\omega) = r_n(\omega)$.

Lemma. Let m be a positive integer and $a = \frac{k}{n}$ where k, n are positive integers, $k < 2n$. Let the greatest common divisor of k and $2n$ be equal to 1. Then for $\omega \in \Omega$ it holds $\sum_{j=1}^{2m\cdot n} T_a^j \omega =$ $= m(2n \cdot r_n(\omega) - 1)$.

Proof. For $\omega \in \Omega$, $T_a \omega, \dots, T_a^{2n}\omega$ differ mutually and $\omega = T_a^{2n}\omega$. Therefore, $\sum_{j=1}^{2n} T_a^j \omega = \sum_{j=0}^{n-1} ((-1 + \frac{j}{n} + r_n(\omega)) +$ $+ (1 - \frac{j+1}{n} + r_n(\omega)) = 2n \cdot r_n(\omega) - 1$. From $\omega = T_a^{2n}\omega$ we get that $\sum_{j=1}^{2m\cdot n} T_a^j \omega = m \cdot \sum_{j=1}^{2n} T_a^j \omega$ which finishes the proof.

Theorem 1. There exists a real number a, $0 < a < 2$, and an increasing sequence $(n_j;\ j = 1,2,\dots)$ of positive integers such that for $j \longrightarrow \infty$ the distributions of $\frac{1}{n_j} \sum_{j=1}^{2n_j^2} I_d \circ T_a^j$ (where I_d is the identity mapping of Ω onto Ω) converge weakly to the uniform distribution on $(-1,1)$.

Proof. Let k_1 and n_1 be any two positive integers such that $k_1 \leq n_1$ and the greatest common divisor of k_1 and $2n_1$ is equal to 1. We define $a_1 = \frac{k_1}{n_1}$, $n_2 = 2k_1 \cdot n_1^4$ and $k_2 = 2k_1 \cdot n_1^3 + 1$. Thus, it is $\frac{k_1}{n_1} + \frac{1}{n_2} = \frac{k_2}{n_2}$.

We can easily convince ourselves that the greatest common divisor of k_2 and $2n_2$ is equal to 1.

In the same way as we have derived k_2 and n_2 from k_1 and n_1, we derive also k_{j+1} and n_{j+1} from k_j, n_j and set $a_{j+1} = \frac{k_{j+1}}{n_{j+1}}$, $j =$ $= 2,3,\dots$. In this way we obtain the numbers $a_j = \frac{k_j}{n_j} =$ $= \frac{k_1}{n_1} + \sum_{i=2}^{j} \frac{1}{n_i}$, $j = 1,2,\dots$, where the greatest common divisor of k_j and $2n_j$ is equal to 1.

The sum $a = \frac{k_1}{n_1} + \sum_{i=2}^{\infty} \frac{1}{n_i}$ is finite and $a_j \xrightarrow[j\to\infty]{} a$. By the Lemma, for any positive integer j, the sum $\frac{1}{n_j} \sum_{i=1}^{2n_j^2} I_d \circ T_{a_j}^i$ has the uniform distribution on (-1.1). Let $L_j = \{\omega : \frac{1}{n_j^2} < r_{n_j}(\omega) < \frac{1}{n_j} - \frac{1}{n_j^2}\}$, $j = 1,2,\dots$. Evidently, $\mu L_j = 1 - \frac{2}{n_j}$. For $\bar{a}_j = a - a_j$ we have $\bar{a}_j = \sum_{i=j+1}^{\infty} \frac{1}{n_i} \le \frac{1}{k_j \cdot n_j^4}$. Assuming $k_j > 2$ we get that for $\omega \in L_j$ and $1 \le i \le 2n_j^2$ it is $|T^i_{a_j}\omega - T^i_a\omega| \le \frac{2}{k_j \cdot n_j^2}$.

Thus, $|\sum_{i=1}^{2n_j^2} T^i_{a_j}\omega - \sum_{i=1}^{2n_j^2} T^i_a\omega| \le \frac{4}{k_j}$. Therefore,

$$\mu\{|\frac{1}{n_j}\sum_{i=1}^{2n_j^2} I_d \circ T^i_{a_j} - \frac{1}{n_j}\sum_{i=1}^{2n_j^2} I_d \circ T^i_a| > \frac{1}{n_j \cdot k_j}\} \xrightarrow[j\to\infty]{} 0.$$

Hence, we obtain that the measures $\mu(\frac{1}{n_j}\sum_{i=1}^{2n_j^2} I_d \circ T^j_a)^{-1}$ converge weakly to the uniform distribution on (-1,1).

Let us suppose that the number a is not irrational. Then for some positive integers n, k, $k < 2n$, we have $a = \frac{k}{n}$. According to the Lemma, the sum $\frac{1}{m \cdot n}\sum_{i=1}^{2m^2 n^2} I_d \circ T^i_a$ has the uniform distribution on (-m,m) for any positive integer m. This contradicts the fact that the measures $\mu(\frac{1}{n_j}\sum_{i=1}^{2n_j^2} I_d \circ T^i_a)^{-1}$ converge weakly to the uniform distribution on (-1,1). This completes the proof.

Let us assign $\mathcal{B}_1$ the σ-algebra of Borel sets on the real line $\mathbb{R}$.

Theorem 2. Let ν be a probability measure on $(\mathbb{R}, \mathcal{B}_1)$ which is absolutely continuous with respect to the Lebesgue measure m with density function g.
If the function g is symmetric and nonincreasing on $\langle 0,\infty)$ then there exists a dynamical system $(\Omega, \mathcal{A}, T, \mu)$ of zero entropy, an increasing sequence $(n_j;\ j = 1,2,\dots)$ of positive integers and a measurable function f on Ω such that the distributions

of $\frac{1}{n_j}\sum_{i=1}^{2n_j^2} f\circ T^i$ converge weakly to ν.

Proof. Let $(\Omega',\mathcal{B}',\mu')$ be the probability space used in the previous sections (i.e. $\Omega' = \langle -1,1)$, $\mathcal{B}'$ is the σ-algebra of Borel sets on Ω' and $\mu' = \frac{1}{2}m$). In accordance with the assumptions of the theorem $g(0) = \sup g$. For $y \in \langle 0,g(0)\rangle$ let us define $h(y) = \sup \{x : g(x) = y\}$. According to the Fubini theorem we have $\int_0^{g(0)} 2h(y)\,dy = \int_{-\infty}^{\infty} g(t)\,dt = 1$ (thus $g(0) > 0$). Let $(\Omega,\mathcal{A},\mu)$ be the product of probability spaces $(\Omega',\mathcal{B}',\mu')$ and $(\Omega'',\mathcal{B}'',\mu'')$ where $\Omega'' = \langle 0,g(0)\rangle$, $\mathcal{B}''$ is the σ-algebra of Borel sets on Ω'' and $\mu''A = \int_A 2h(y)\,dy$ for $A \in \mathcal{B}''$.

Let $a \in (0,2)$ be from Theorem 1. For $(x,y) \in \Omega$ we define $T(x,y) = (\psi(x+a),y)$; then $(\Omega,\mathcal{A},T,\mu)$ is a dynamical system. Let us define probability measures $\mu_y : A \longmapsto \mu'\{x : (x,y) \in A\}$, $y \in \Omega''$ on the measure space $(\Omega,\mathcal{B})$. It holds that $\mu A = \int \mu_y A \, d\mu''(y)$, $A \in \mathcal{B}$, and for each $y \in \Omega''$ the dynamical system $(\Omega,\mathcal{A},T,\mu_y)$ is isomorphic to the system used in Theorem 1 (the measures μ_y, $y \in \Omega''$, are the ergodic parts of μ, compare [4]).

On the set Ω let us define a function $f : (x,y) \longmapsto x\cdot h(y)$. For a real number z and for $y \in \Omega''$ let us set $F_y(z) = \mu_y\{x : f(x,y) < z\}$. According to the Fubini theorem, $\mu\{\omega \in \Omega : f(\omega) < z\} = \int_0^{g(0)} 2h(y)\,F_y(z)\,dy = \int_0^{g(0)} \left(\int_{-\infty}^{z} \chi_{\langle -h(y),h(x)\rangle}(x)\,dx\right)dy = \int_{-\infty}^{z} g(t)\,dt$. Hence we obtain that $\nu = \mu f^{-1}$.

Let us assign $s_j = \frac{1}{n_j}\sum_{i=1}^{2n_j^2} f\circ T^i$, $j = 1,2,\dots$. By Theorem 1, the measures $\mu_y\left(\frac{1}{n_j}\sum_{i=1}^{2n_j^2} f\circ T^i\right)^{-1}$ converge weakly to the uniform distribution on $(-h(y),h(y))$, i.e. to $\mu_y f^{-1}$.

For $y \in \Omega''$ and $j \in \{1,2,\dots\}$ let $\varphi_y^{(j)}$ be the characteristic function of the measure $\mu_y s_j^{-1}$ and let φ_y be the characteristic function of the measure $\mu_y f^{-1}$. It holds that $\varphi_y^{(j)} \xrightarrow[j \to \infty]{} \varphi_y$ (uniformly on each compact subset of $\mathbb{R}$). Let us denote $\varphi^{(j)}$ the characteristic function of μs_j^{-1} and φ the characteristic function of μf^{-1}. Evidently, it is $\varphi^{(j)} = \int \varphi_y^{(j)} d\mu'' (y)$ and $\varphi = \int \varphi_y \, d\mu'' (y)$. It holds that $\varphi^{(j)} \longrightarrow \varphi$. Hence we obtain that the measures μs_j^{-1} converge weakly to ν which finishes the proof of the theorem.

Remark. If $b = \int g(\sqrt[3]{|x|})dx < \infty$ then $\int f^2 d\mu \le \int 2h(y) \cdot h^2(y)dy) = b < \infty$ and $f \in L^2(\mu)$.

R e f e r e n c e s

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Matematický ústav, Univerzita Karlova, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

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