COMMENTATIONES MATHEMATICAE UNVERSITATIS CAROLINAE

26,2 (1985)

NONISOMORPHIC THIN-TALL SUPERATOMIC BOOLEAN ALGEBRAS

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Abstract: We shall construct a thin-tall space X satisfying the following: Whenever $\{M,N\}$ covers the set of all isolated points of X and $|\overline{M}| = |\overline{N}| = \omega_1$, then $|\overline{M} \cap \overline{N}| = \omega_1$, too. This in turn implies that there are nonhomeomorphic thin-tall spaces, since not all thin-tall spaces have the above property.

Key words: Scattered space, superatomic Boolean algebra, thin-tall space, thin-tall sBA, Stone duality.

Classification: Primary 54D30, 54G20
Secondary 54A25, 06E15

First, let us recollect the basic notions.

All spaces are assumed to be Boolean, i.e. compact, Hausdorff and zero-dimensional.

A space is called scattered if each nonempty subspace has an isolated point. If X is a scattered space, denote $X = X_0$, $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ for a limit ordinal α , $X_{\alpha+1}$ = the set of all non-isolated points of X_α. The Cantor-Bendixson height of a scattered space X, $ht(X) = \min \{\alpha : X_\alpha = \emptyset\}$.

The compactness of X implies that ht(X) is always a successor ordinal and $X_{ht(X)-1}$ is finite. For $\alpha < ht(X)$, denote $At(X_\alpha) = X_\alpha - X_{\alpha+1}$, the set of all isolated points of X_α. The width of a scattered space X, wd(X), is then $\sup \{|At(X_\alpha)|, \alpha < ht(X)\}$. A space X is called <u>thin-tall</u>, provided it is scattered, $ht(X) = \omega_1 + 1$, $wd(X) = \omega$, $|X_{\omega_1}| = 1$.

Indeed, there is a Boolean counterpart to the notions just mentioned. A Boolean algebra $\mathcal{B}$ is superatomic provided that each homomorphic image of $\mathcal{B}$ has atoms; denote $\mathcal{J}_0 = \emptyset$, $\mathcal{J}_\alpha = \bigcup_{\beta<\alpha} \mathcal{J}_\beta$ for a limit ordinal α, let $\mathcal{J}_{\alpha+1}$ be the ideal generated by $\mathcal{J}_\alpha$ and the set of all atoms of $\mathcal{B}/\mathcal{J}_\alpha$. Then $ht(\mathcal{B}) = \min\{\alpha : \mathcal{J}_\alpha = \mathcal{B}\}$ and $wd(\mathcal{B}) = \sup\{|At(\mathcal{B}/\mathcal{J}_\alpha)| : \alpha < ht(\mathcal{B})\}$, where $At(\mathcal{B}/\mathcal{J}_\alpha)$ is the set of all atoms of $\mathcal{B}/\mathcal{J}_\alpha$. A superatomic Boolean algebra (abbr. sBA) is called thin-tall, if its height is ω_1 and its width equals ω.

The classical result of Mazurkiewicz and Sierpiński [MS] says that any two countable superatomic BA's are isomorphic provided that they have the same height.

The things are different in the case of thin-tall sBA's. It was the second author of the present paper, who showed in [W] that there are nonisomorphic thin-tall sBA's under the assumption of CH. For this purpose, he constructed a thin-tall space X such that each autohomeomorphism of X moves countably many points at most. Though the existence of this kind of spaces is still open in ZFC, the different approach enabled us to remove CH from the main result.

<u>Theorem 1</u>. There are nonhomeomorphic thin-tall spaces.

Here we shall adopt the different way of reasoning. The reader can recognize that the basic idea goes back to Luzin [L]. Instead of counting homeomorphisms, we derive Theorem 1 from the forthcoming

<u>Theorem 2</u>. There is a thin-tall space X satisfying the

following: If $M \cup N = At(X)$ and $|\overline{M}| = |\overline{N}| = \omega_1$, then $|\overline{M} \cap \overline{N}| = \omega_1$, too.

Indeed, if Y is an arbitrary thin-tall space, then the quotient space $T = Y \times \{0,1\}/\sim$, where $x \sim y$ for $x \neq y$ only if $x = (z,0)$, $y = (z,1)$ and $\{z\} = Y_{\omega_1}$, is thin-tall, too, nevertheless T does not have the property described in Theorem 2: Consider $M = At(Y) \times \{0\}$, $N = At(Y) \times \{1\}$.

Theorem 1 being proved, it remains to prove Theorem 2.

We shall apply the standard trick. Instead of looking for the space X we shall find a sBA $\mathcal{B}$ such that X will be its Stone space. And rather than to construct the whole of $\mathcal{B}$, we shall determine the set of its generators, which is often called a representation sequence. Since we may w.l.o.g. assume that $\mathcal{B}$ is a subalgebra of $\mathcal{P}(\omega)$, the representation sequence is the family

$$\{R_{\alpha,n} : \alpha < \omega_1, n < \omega\} \subseteq \mathcal{P}(\omega)$$

satisfying (0) - (3) below.

(0) $R_{0,n} = \{n\}$ for all $n < \omega$;

(1) $R_{\alpha,n} \cap R_{\alpha,m} = \emptyset$ for each $\alpha < \omega_1$, $n < m < \omega$.

Denote $\mathcal{J}_0 = \emptyset$, $\mathcal{J}_\alpha = \bigcup_{\beta < \alpha} \mathcal{J}_\beta$ for a limit $\alpha < \omega_1$, and $\mathcal{J}_{\alpha+1} = \mathcal{J}_\alpha \cup \{M \subseteq \omega : (\exists F \in [\omega]^{<\omega})\ M \subseteq \bigcup_{m \in F} R_{\alpha,n}\}$.

(2) For each $\beta < \alpha < \omega_1$ and for each $n, m \in \omega$, either $R_{\beta,n} \cap R_{\alpha,m} \in \mathcal{J}_\beta$ or $R_{\beta,n} - R_{\alpha,m} \in \mathcal{J}_\beta$;

(3) for each $\beta < \alpha < \omega_1$ and for each $m \in \omega$,

$$|\{n \in \omega : R_{\beta,n} - R_{\alpha,m} \in \mathcal{J}_\beta\}| = \omega$$

The reader is invited to check that any thin-tall sBA $\mathcal{B} \subseteq \mathcal{P}(\omega)$ is generated by a suitable family $\{R_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ satisfying (0) - (3), and vice versa, any family

$\{R_{\alpha,n}: \alpha < \omega_1, n < \omega\}$ satisfying (0) - (3) generates a subalgebra of $\mathcal{P}(\omega)$ which is superatomic and thin-tall.

Let a thin-tall sBA $\mathcal{B} \subseteq \mathcal{P}(\omega)$ be generated by $\{R_{\alpha,n}: \alpha < \omega_1, n < \omega\}$. The forthcoming description of the Stone space of $\mathcal{B}$ is also very simple and the reader can verify it after a moment of reflection: For $\alpha < \omega_1$, $n < \omega$ denote $\mathcal{F}_{\alpha,n}$ the filter on ω generated by $\{R_{\alpha,n} - Q: Q \in \mathcal{J}_\alpha\}$. Then $St(\mathcal{B})$ is homeomorphic to the quotient space $\beta\omega/\sim$, where the equivalence relation $\sim$ is defined as follows: For $p,q \in \beta\omega$, $p \sim q$ iff ($p \supseteq \mathcal{F}_{\alpha,n}$ if and only if $q \supseteq \mathcal{F}_{\alpha,n}$ for each $\alpha < \omega_1$, $n < \omega$). Similar description is used in [R].

We shall construct the desired representing sequence by an induction to ω_1. According to the previous, the sequence must satisfy (0) - (3), but we shall want it to satisfy more. The first additional requirement is of the technical nature.

(4) For each $\alpha < \omega_1$ and for each $n,m < \omega$,

$|\{\beta < \alpha : R_{\beta,n} - R_{\alpha,m} \in \mathcal{J}_\beta\}| < \omega$.

The possibility to pass with the transfinite induction through is the statement of our first lemma. To make the life easier, if (k) is any of our conditions and if $\gamma < \omega_1$, then the condition obtained by substituting γ in each occurence of ω_1 in (k) will be denoted by $(k)_\gamma$.

<u>Lemma 1.</u> Let $\gamma < \omega_1$, let $\{R_{\alpha,n}: \alpha < \gamma, n < \omega\}$ satisfy $(0)_\gamma - (4)_\gamma$. Then there is a family $\{R_{\gamma,n}: n < \omega\}$ such that $(0)_{\gamma+1} - (4)_{\gamma+1}$ holds.

□ Case $\gamma = \alpha + 1$.

Choose an arbitrary partition $\{Z_m: m < \omega\}$ of ω such that each Z_m is infinite and enumerate $\{R_k: k < \omega\}$ the set $\{R_{\beta,n}:$

: $\beta < \alpha, n < \omega\}$. Denote $S_{\alpha,n} = R_{\alpha,n} - \bigcup\{R_{\beta,i}: \beta < \alpha \& i \le n \& \& R_{\beta,i} - R_{\alpha,n} \in \mathcal{J}_\beta\} - \bigcup\{R_k : k \le n\}$. Define then $R_{\gamma,m} = \bigcup\{S_{\alpha,n} : n \in Z_m\}$. We have to verify that this works.

$(0)_{\gamma+1}$ is clear, $(1)_{\gamma+1}$ follows by the fact that $\{R_{\alpha,n} : n < \omega\}$ is pairwise disjoint, consequently $\{S_{\alpha,n} : n < \omega\}$ is, and by the disjointness of $\{Z_m : m < \omega\}$.

Let us verify $(2)_{\gamma+1}$. According to our definition, $R_{\alpha,n} - S_{\alpha,n} \in \mathcal{J}_\alpha$, therefore $R_{\alpha,n} - R_{\gamma,m} \in \mathcal{J}_\alpha$ or $R_{\alpha,n} \cap R_{\gamma,m} \in \mathcal{J}_\alpha$.

If $\beta < \alpha$, $n \in \omega$, then for some $k < \omega$, $R_{\beta,n} = R_k$ in our enumeration. The definition of $S_{\alpha,j}$ implies that $R_{\beta,n} \cap S_{\alpha,j} \neq \emptyset$ only if $j < k$. By $(1)_\gamma$, $(2)_\gamma$, there is at most one $j_0 < \omega$ such that $R_{\beta,n} - R_{\alpha,j_0} \in \mathcal{J}_\beta$. Thus $R_{\beta,n} \cap \bigcup\{S_{\alpha,j} : j < \omega, j \neq j_0\} \in \mathcal{J}_\beta$, because $R_{\beta,n} \cap \bigcup\{S_{\alpha,j} : j < \omega, j \neq j_0\} \subseteq R_{\beta,n} \cap \bigcup\{S_{\alpha,j} : j < k, j \neq j_0\} \subseteq R_{\beta,n} \cap \bigcup\{R_{\alpha,j} : j < k, j \neq j_0\} = \bigcup\{R_{\beta,n} \cap R_{\alpha,j} : j < k, j \neq j_0\}$, which is a finite union of members of $\mathcal{J}_\beta$.

Consequently $R_{\beta,n} \cap R_{\gamma,m} \in \mathcal{J}_\beta$ for each $m < \omega$ such that $j_0 \notin Z_m$, and if $j_0 \in Z_m$, but $j_0 \ge k$, then $R_{\beta,n} \cap R_{\gamma,m} \in \mathcal{J}_\beta$, too.

So suppose $j_0 \in Z_m$ and $j_0 < k$. Denote by $\mathcal{R}$ the family $\{R_i : i \le j_0\} \cup \{R_{\delta,i} : \delta < \alpha \& i \le j_0 \& R_{\delta,i} - R_{\alpha,j_0} \in \mathcal{J}_\delta\}$. $\mathcal{R}$ is finite, $S_{\alpha,j_0} = R_{\alpha,j_0} - \bigcup\mathcal{R}$, and if for each $R \in \mathcal{R}$, $R_{\beta,n} \cap R \in \mathcal{J}_\beta$, then $R_{\beta,n} - S_{\alpha,j_0} \subseteq (R_{\beta,n} - R_{\alpha,j_0}) \cup \bigcup\{R_{\beta,n} \cap R : R \in \mathcal{R}\}$, which belongs to $\mathcal{J}_\beta$, hence $R_{\beta,n} - R_{\gamma,m} \in \mathcal{J}_\beta$, too.

If for some $\tilde{R} \in \mathcal{R}$, $R_{\beta,n} \cap \tilde{R} \notin \mathcal{J}_\beta$, then by $(2)_\gamma$, $R_{\beta,n} - \tilde{R} \in \mathcal{J}_\beta$. Thence $R_{\beta,n} \cap S_{\alpha,j_0} \subseteq R_{\beta,n} \cap (R_{\alpha,j_0} - \tilde{R}) \subseteq R_{\beta,n} - \tilde{R} \in \mathcal{J}_\beta$. We have $R_{\beta,n} \cap R_{\gamma,m} \in \mathcal{J}_\beta$ in this case. So $(2)_{\gamma+1}$ is satisfied.

$(3)_{\gamma+1}$ follows easily by (3) and by the fact that each Z_m was chosen to be infinite.

Finally, if $n \leq j$ and if $R_{\beta,n} - R_{\alpha,j} \in \mathcal{J}_\beta$, then $R_{\beta,n} \cap \cap S_{\alpha,j} = \emptyset$ according to the definition of $S_{\alpha,j}$. Thus for each $n,m < \omega$, the set $\{\beta < \gamma : R_{\beta,n} - R_{\gamma,m} \in \mathcal{J}_\beta\} \subseteq \{\alpha\} \cup \{\beta < \gamma : (\exists j < n)\ R_{\beta,n} - R_{\alpha,j} \in \mathcal{J}_\beta\} = \{\alpha\} \cup \bigcup_{j=0}^{n-1} \{\beta < \gamma : R_{\beta,n} - R_{\alpha,j} \in \mathcal{J}_\beta\}$, which is a finite union of sets, each of them being finite by $(4)_\gamma$. Thus $(4)_{\gamma+1}$ holds, too.

Case γ is a limit ordinal.

Pick a sequence $\gamma_\ell \nearrow \gamma$ ($\ell < \omega$). For each $\ell < \omega$, let $\{R_k^\ell : k < \omega\}$ be an enumeration of $\{R_{\beta,n} : \beta < \gamma_\ell, n < \omega\}$. Let $T_{\gamma_\ell,n} = R_{\gamma_\ell,n} - \bigcup\{R_{\beta,i} : \beta < \gamma_\ell \ \&\ i \leq n \ \&\ R_{\beta,i} - R_{\gamma_\ell,n} \in \mathcal{J}_\beta\} - \bigcup\{R_k^j : k \leq n, j \leq \ell\}$.

Let $S_{0,0} = T_{\gamma_0,0}$, then define $S_{\ell,m}$ by an induction as follows: for $m \leq \ell < \omega$, let

$$S_{\ell,m} = T_{\gamma_\ell, g(\ell)+m} - \bigcup\{S_{i,k} : i < \ell, k \leq i\}$$

where $g(\ell)$ is a natural number such that $g(\ell) \geq \ell$ and for each $n \geq g(\ell)$, $R_{\gamma_\ell,n} \cap S_{i,k} \in \mathcal{J}_{\gamma_i}$ whenever $i < \ell$, $k \leq i$. The assumptions $(2)_\gamma$, $(3)_\gamma$ guarantee the existence of $g(\ell)$.

Finally, let $R_{\gamma,m} = \bigcup_{\ell=m}^{\infty} S_{\ell,m}$.

Now we have to verify that $(0)_{\gamma+1} - (4)_{\gamma+1}$ again hold. But - modulo the more involved notation - one can step by step mimick the corresponding parts from the previous case, so we leave it to the reader. □

Next comes the essential step of our construction. We shall find sets $A_{\alpha,n}$, which will serve as a major tool for the proof of Theorem 2. Their properties are listed below:

(5) $A_{\alpha,n} \subseteq R_{\alpha,n}$ for each $1 \le \alpha < \omega_1$, and each $n < \omega$;

(6) for each $n < \omega$ and each $1 \le \beta < \alpha < \omega_1$,

$|\{R_{\beta,n} \cap \bigcup\{A_{\alpha,m}: m < \omega\}| < \omega$;

(7) for each $1 \le \alpha < \omega_1$ and for each $m, p < \omega$,

$|\{(\beta, n): \beta < \alpha \ \& \ n \le m \ \& \ |A_{\beta,n} \cap A_{\alpha,m}| \le p\}| < \omega$.

Our aim is to show that there are $\{R_{\alpha,n}: \alpha < \omega_1, n < \omega\}$ and $\{A_{\alpha,n}: 1 \le \alpha < \omega_1, n < \omega\}$ such that (0) - (7) hold for them. This will be done in the next two lemmas.

Lemma 2. Let $1 \le \gamma < \omega_1$ and suppose that $\{R_{\alpha,n}: \alpha < \gamma, n < \omega\}$ and $\{A_{\alpha,n}: 1 \le \alpha < \gamma, n < \omega\}$ satisfy $(0)_\gamma$ - $(7)_\gamma$. Then there is a pairwise disjoint family $\{A_{\gamma,n}: n < \omega\}$ such that $(6)_{\gamma+1}$ and $(7)_{\gamma+1}$ hold.

□ If $\gamma = 1$, there is nothing to prove: let $\{A_{1,n}: n < \omega\}$ be an arbitrary partition of ω into infinite sets.

So suppose $\gamma > 1$ and let $\{R_k: 1 \le k < \omega\}$ and $\{A_k: 1 \le k < \omega\}$ be an enumeration of $\{R_{\alpha,n}: 1 \le \alpha < \gamma, n < \omega\}$ and of $\{A_{\alpha,n}: 1 \le \alpha < \gamma, n < \omega\}$ such that $A_k = A_{\alpha,n}$ iff $R_k = R_{\alpha,n}$ for each natural k and n, $1 \le \alpha < \gamma$. Using an induction, we shall define sets $P_{k,m}$ and families $\mathcal{C}(k)$ for $k < \omega$, $m \le k$ as follows.

$$\mathcal{C}(0) = \emptyset, \ P_{0,0} = \emptyset.$$

Suppose $\mathcal{C}(i)$ and $P_{i,m}$ are known for $i < k$, $m \le i$. Our inductive assumptions are:

(a) each $P_{i,m}$ is a finite subset of $\bigcup\mathcal{C}(i) - \bigcup\mathcal{C}(i-1)$,

(b) each $\mathcal{C}(i)$ is finite,

(c) $\mathcal{C}(i-1) \subseteq \mathcal{C}(i) \subseteq \{R_{\alpha,n}: 1 \le \alpha < \gamma, n < \omega\}$.

Let $R_{\alpha,n} = R_k$. If there is some $R \in \mathcal{C}(k-1)$ with $R_{\alpha,n}$ -

$- R \in \mathcal{J}_\alpha$, define $\mathcal{C}(k) = \mathcal{C}(k-1)$, $P_{k,m} = \emptyset$ for all $m \leq k$.

In the opposite case, for each $R \in \mathcal{C}(k-1)$, $R \cap R_{\alpha,n} \in \mathcal{J}_\alpha$ by $(2)_\gamma$. We shall find the family $\mathcal{C}(k)$ first, and to do this, we need an induction again.

Let $\mathcal{C}_0(k) = \mathcal{C}(k-1) \cup \{R_k\}$, further, let $\mathcal{C}_{s+1}(k) = \{R_{\beta,j} : 1 \leq \beta < \gamma \ \& \ j \leq k \ \& \ \exists R \in \mathcal{C}_s(k)$ with $R_{\beta,j} - R \in \mathcal{J}_\beta\} - \mathcal{C}_s(k)$, $\mathcal{C}(k) = \bigcup_{s=0}^{\infty} \mathcal{C}_s(k)$.

<u>Claim</u>. $\mathcal{C}(k)$ is finite. Indeed, by $(4)_\gamma$ and by $|\mathcal{C}(k-1)| < \omega$, $\mathcal{C}_s(k) < \omega$ for each $s < \omega$. Define $\beta(s) = \max\{\beta < \gamma : \exists j < \omega$ with $R_{\beta,j} \in \mathcal{C}_s(k)\}$. If $R_{\beta,j} - R_{\delta,t} \in \mathcal{J}_\beta$ and $(\beta, j) \neq (\delta, t)$, then $\beta < \delta$ by $(2)_\gamma$ and $(3)_\gamma$, hence $\beta(s) > \beta(s+1)$ if $\mathcal{C}_s(k)$ and $\mathcal{C}_{s+1}(k)$ are nonempty. Therefore $\mathcal{C}_s(k) \neq \emptyset$ for finitely many indices s only, which proves the claim.

Let $M_k = \bigcup \mathcal{C}(k) - \bigcup \mathcal{C}(k-1)$, let $\tilde{\mathcal{C}}(k) = \{R_{\beta,j} \in \mathcal{C}(k) : \forall R \in \mathcal{C}(k-1),\ R_{\beta,j} \cap R \in \mathcal{J}_\beta\}$, let $I = \{\ell < \omega : R_\ell \in \tilde{\mathcal{C}}(k)\}$.

For $\ell \in I$ we have $|A_\ell \cap M_k| = \omega$ by $(2)_\gamma$, $(5)_\gamma$ and $(6)_\gamma$, hence we can find sets $P_k(\ell, m)$ $(\ell \in I,\ m \leq k)$ such that $|P_k(\ell, m)| = k$, $P_k(\ell, m) \subseteq A_\ell \cap M_k$, $P_k(\ell, m) \cap P_k(\ell', m') = \emptyset$ whenever $(\ell, m) \neq (\ell', m')$.

It remains to define $P_{k,m} = \bigcup \{P_k(\ell, m) : \ell \in I\}$. This completes the inductive definition.

As may be expected, we set $A_{\gamma,m} = \bigcup_{k=m}^{\infty} P_{k,m}$.

Clearly $A_{\gamma,m} \cap A_{\gamma,m'} = \emptyset$ for $m \neq m'$. We have to verify $(6)_{\gamma+1}$ and $(7)_{\gamma+1}$.

Let $1 \leq \beta < \gamma$ and $n < \omega$ be arbitrary, we have to check that $R_{\beta,n} \cap \bigcup_{m<\omega} A_{\gamma,m}$ is finite. It suffices to show that for some $k < \omega$, $R_{\beta,n} \in \mathcal{C}(k)$: Indeed, if $R_{\beta,n} \in \mathcal{C}(k)$, then by (a) and (c) of our construction, $R_{\beta,n} \cap \bigcup_{m<\omega} A_{\gamma,m} \subseteq \bigcup \mathcal{C}(k) \cap$ $\cap \bigcup_{m<\omega} A_{\gamma,m} = \bigcup \{P_{\ell,m} : \ell \leq k, m \leq \ell\}$, and the last set is finite.

Let $R_{\beta,n} = R_t$ in our enumeration. If for each $R \in \mathcal{C}(t-1)$, $R_{\beta,n} \cap R \in \mathcal{J}_\beta$, then $R_{\beta,n} \in \mathcal{C}_0(t) \subseteq \mathcal{C}(t)$ and we are done. If for some $R \in \mathcal{C}(t-1)$, $R_{\beta,n} - R \in \mathcal{J}_\beta$, pick the first $k \geq \max(n,t) + 1$ such that $\mathcal{C}(k) \neq \mathcal{C}(k-1)$. The existence of such a k follows by $(1)_\gamma$, $(2)_\gamma$ and $(3)_\gamma$. The definition of $\mathcal{C}(k)$ guarantees that $R_{\beta,n} \in \mathcal{C}_1(k) \subseteq \mathcal{C}(k)$. Thus $(6)_{\gamma+1}$ holds.

For $(7)_{\gamma+1}$, it suffices to show that for each $n \leq m < \omega$ and for each $p < \omega$, $|\{\beta < \gamma : |A_{\beta,n} \cap A_{\gamma,m}| \leq p\}| < \omega$.

Let $k = \max(p,n,m) + 1$ and consider the family $\mathcal{C}(k-1)$. Since $\mathcal{C}(k-1)$ is finite, by $(4)_\gamma$ the set $\Gamma = \{\beta < \gamma :$ For some $R \in \mathcal{C}(k-1)$, $R_{\beta,n} - R \in \mathcal{J}_\beta\}$ is finite. We claim that for each $\beta < \gamma$, if $\beta \notin \Gamma$, then $|A_{\beta,n} \cap A_{\gamma,m}| > p$.

Choose such a β and let $R_{\beta,n} = R_\ell$ in our enumeration. Since $\beta \notin \Gamma$, $R_{\beta,n} \in \mathcal{C}(k') - \mathcal{C}(k'-1)$ for some $k' \geq k$, moreover, $k' > n$ implies that: $R_{\beta,n} \in \tilde{\mathcal{C}}(k')$ then. Now, by the inductive definition, we have $|A_{\gamma,m} \cap A_{\beta,n}| \geq |P_{k'}(\ell, m)| = k' \geq k > p$.

The lemma is proved. □

<u>Lemma 3</u>. Let $1 \leq \gamma < \omega$ and let $\{R_{\alpha,n} : \alpha < \gamma, n < \omega\}$ and $\{A_{\alpha,n} : 1 \leq \alpha < \gamma, n < \omega\}$ satisfy $(0)_\gamma - (7)_\gamma$. Then there are $\{R_{\gamma,n} : n < \omega\}$ and $\{A_{\gamma,n} : n < \omega\}$ such that $(0)_{\gamma+1} - (7)_{\gamma+1}$ hold.

□ Applying Lemma 2, we obtain the collection $\{A_{\gamma,n} : n < \omega\}$. For each $\alpha < \gamma$, let $R'_{\alpha,n} = R_{\alpha,n} - \bigcup_{m<\omega} A_{\gamma,m}$. Since $(6)_{\gamma+1}$ holds, it is clear that $\{R'_{\alpha,n} : \alpha < \gamma, n < \omega\}$ - when viewed as a collection of subsets of $\{n \in \omega : R'_{0,n} \neq \emptyset\}$ rather than of

ω - satisfies $(0)_\gamma$ - $(4)_\gamma$. Using Lemma 1, find a family $\{R'_{\gamma,n}: n < \omega\}$ such that $(1)_{\gamma+1}$ - $(4)_{\gamma+1}$ holds for $\{R'_{\alpha,n}: \alpha < \gamma + 1, n < \omega\}$. We may and shall assume that each $R'_{\gamma,n}$ is disjoint with $\bigcup_{m<\omega} A_{\gamma,m}$.

Let $R_{\gamma,n} = R'_{\gamma,n} \cup A_{\gamma,n}$. The validity of $(0)_{\gamma+1}$ - $(7)_{\gamma+1}$ for $\{R_{\alpha,n}: \alpha < \gamma + 1, n < \omega\}$ and $\{A_{\alpha,n}: 1 \leq \alpha < \gamma + 1, n < \omega\}$ is obvious. $\square$

Having established the necessary lemmas, we know that there are $\{R_{\alpha,n}: \alpha < \omega_1, n < \omega\} \subseteq \mathcal{P}(\omega)$ and $\{A_{\alpha,n}: 1 \leq \alpha < \omega_1, n < \omega\} \subseteq \mathcal{P}(\omega)$ satisfying (0) - (7): The transfinite induction was proved to work.

Let $\mathcal{B}$ be a subalgebra of $\mathcal{P}(\omega)$ generated by $\{R_{\alpha,n}: \alpha < \omega_1, n < \omega\}$, X its Stone space. The conditions (0) - (3) make X to be thin-tall.

We have already described X as a quotient space $\beta\omega/\sim$. Using this description, denote $x_{\alpha,n}$ the point of X corresponding to $\mathcal{F}_{\alpha,n}$. Remind that $\mathcal{F}_{\alpha,n}$ is the filter on ω generated by $\{R_{\alpha,n} - Q: Q \in \mathcal{J}_\alpha\}$. Of course one identifies ω with the set of all isolated points of X.

Notice that for each $1 \leq \alpha < \omega_1$ and each $n < \omega$, the set $A_{\alpha,n}$ converges to a point $x_{\alpha,n}$: By (5), $A_{\alpha,n} \subseteq R_{\alpha,n}$, and by (6), if $Q \in \mathcal{J}_\alpha$, then $A_{\alpha,n} \cap Q$ is finite. Thus $A_{\alpha,n} - (R_{\alpha,n} - Q)$ is finite, but this means that arbitrary neighborhood of $x_{\alpha,n}$ contains all but finitely many points of $A_{\alpha,n}$.

Let $M \subseteq \omega$ be such that $|\overline{M}| = \omega_1 = |\overline{\omega - M}|$. We have to show that $|\overline{M} \cap (\overline{\omega - M})| = \omega_1$, too.

Suppose not. Then there is some $\alpha_0 < \omega_1$ such that for each $\alpha > \alpha_0$ and each $n < \omega$, $x_{\alpha,n} \notin \overline{M} \cap (\overline{\omega - M})$. Let

$I = \{\alpha > \alpha_0$: for some $n < \omega$, $x_{\alpha,n} \in \overline{M}\}$,
$J = \{\alpha > \alpha_0$: for some $n < \omega$, $x_{\alpha,n} \in (\overline{\omega - M})\}$.
Clearly $|I| = |J| = \omega_1$ because $|\overline{M}| = |\overline{\omega - M}| = \omega_1$. Since ω is countable and I as well as J is uncountable, there are some $\bar{n}, \bar{m} < \omega$ such that both $I' = \{\alpha \in I: x_{\alpha,\bar{n}} \in \overline{M}\}$ and $J' = \{\alpha \in J: x_{\alpha,\bar{m}} \in (\overline{\omega - M})\}$ are uncountable.

If $\alpha \in I'$, then $x_{\alpha,\bar{n}} \notin \overline{\omega - M}$, since $\alpha > \alpha_0$. We have shown that $A_{\alpha,\bar{n}}$ converges to $x_{\alpha,\bar{n}}$, therefore $A_{\alpha,\bar{n}} \cap (\omega - M)$ must be finite for $\alpha \in I'$. Similarly, $A_{\alpha,\bar{m}} \cap M$ is finite for $\alpha \in J'$. Thus there are $p,q < \omega$ such that the set $I'' = \{\alpha \in I': |A_{\alpha,\bar{n}} \cap (\omega - M)| \leq p\}$ as well as $J'' = \{\alpha \in J': |A_{\alpha,\bar{m}} \cap M| \leq q\}$ is uncountable.

We may w.l.o.g. assume that $\bar{n} \leq \bar{m}$. Now, I'' and J'' being uncountable, there is some $\alpha \in J''$ such that $|\{\beta \in I'': \beta < \alpha\}| = \omega$.

For $\beta \in I''$, $\beta < \alpha$ we have:

$$|A_{\beta,\bar{n}} \cap A_{\alpha,\bar{m}}| = |A_{\beta,\bar{n}} \cap A_{\alpha,\bar{m}} \cap (\omega - M)| + |A_{\beta,\bar{n}} \cap A_{\alpha,\bar{m}} \cap M| \leq$$
$$\leq |A_{\beta,\bar{n}} \cap (\omega - M)| + |A_{\alpha,\bar{m}} \cap M| \leq p + q.$$

But this contradicts (7).

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(Oblatum 23.10. 1984)