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MULTI-PHASE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF MOTION OF GENERAL FLUIDS

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Abstract The nonstationary multi-phase free boundary problem for the equations of motion of general fluids is investigated. The proof is given by the well-known theory of parabolic system in Hölder spaces.

Key words. Multi-phase free boundary problem, General fluids.

AMS classification. 35Q99, 76N10, 76T05

1. Introduction.

There are many famous and interesting problems in hydrodynamics, whose outstanding feature is somewhat paradoxical fact that the boundary of the flow is itself not given. While there is a great variety of problems with free boundaries, some of which were already investigated in Newton's day, it seems to the present author that they do study just a little from both a real physical and a strict mathematical point of view.

The one-phase free boundary problems for incompressible viscous fluids are discussed by Solonnikov [5] and Beale [1,2] and those for compressible ones, by Tani [7] and Secchi-Valli [3].

But concerning the multi-phase free boundary problems both for incompressible and compressible viscous fluids there is only one result [8,9], as far as the author knows until now.

In this paper, we confine ourselves to the multi-phase free boundary problem for the system of differential equations of motion of compressible viscous isotropic Newtonian fluids, say general fluids.

Notation. For a domain Ω in $\mathbb{R}^3$, any non-negative integer n and $\alpha \in (0,1)$, we define:

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$$C^{n+\alpha}(\bar\Omega)=\{f(x),\text{ defined on }\bar\Omega \mid \|f\|_{\bar\Omega}^{(n+\alpha)}\equiv \sum_{|s|=0}^{n}|D_x^s f|_{\bar\Omega}^{(0)}+\sum_{|s|=n}|D_x^s f|_{\bar\Omega}^{(\alpha)}<+\infty\},$$

$$C_{x,t}^{n+\alpha,(n+\alpha)/2}(\bar Q_T)=\{g(x,t),\text{ defined on }\bar Q_T\equiv\bar\Omega\times[0,T] \mid \|g\|_{\bar Q_T}^{(n+\alpha)}\equiv$$

$$\equiv\sum_{2r+|s|=0}^{n}|D_t^rD_x^s g|_{\bar Q_T}^{(0)}+\sum_{2r+|s|=n}|D_t^rD_x^s g|_{x,\bar Q_T}^{(\alpha)}+$$

$$+\sum_{2r+|s|=\max(n-1,0)}^{n}|D_t^rD_x^s g|_{\bar Q_T}^{((n-2r-|s|+\alpha)/2)}<+\infty\},$$

$$B^n(\bar Q_T)=\{h(x,t),\text{ defined on }\bar Q_T \mid \sum_{r+|s|=0}^{n}|D_t^rD_x^s h|_{\bar Q_T}^{(0)}<+\infty\},$$

$$B^{n+\alpha}(\bar Q_T)=\{h(x,t),\text{ defined on }\bar Q_T \mid \sum_{r+|s|=0}^{n}|D_t^rD_x^s h|_{\bar Q_T}^{(0)}+\sum_{r+|s|=n}|D_t^rD_x^s h|_{\bar Q_T}^{(\alpha)}<+\infty\},$$

(D_x^s, $D_t^rD_x^s$ and $|s|$ are defined in a conventional way)

$$|f|_{\bar\Omega}^{(0)}=\sup_{\bar\Omega}|f(x)|,\quad |f|_{\bar\Omega}^{(\alpha)}=\sup_{x,x'\in\bar\Omega,x\neq x'}|x-x'|^{-\alpha}|f(x)-f(x')|,$$

$$|g|_{\bar Q_T}^{(0)}=\sup_{\bar Q_T}|g(x,t)|,\quad |g|_{x,\bar Q_T}^{(\alpha)}=\sup_{(x,t),(x',t)\in\bar Q_T,x\neq x'}|x-x'|^{-\alpha}|g(x,t)-g(x',t)|,$$

$$|g|_{t,\bar Q_T}^{(\alpha)}=\sup_{(x,t),(x,t')\in\bar Q_T,t\neq t'}|t-t'|^{-\alpha}|g(x,t)-g(x,t')|,\quad |g|_{\bar Q_T}^{(\alpha)}=|g|_{x,\bar Q_T}^{(\alpha)}+|g|_{t,\bar Q_T}^{(\alpha/2)}.$$

Using local coordinates, it is not difficult to define such spaces for functions defined on the boundary of Ω. The same notations will be used for the spaces of vector functions, whose norms are supposed to be equal to the sum of the norms of all its components. For the Hölder exponent $\alpha=1$, notations such as $|g|_{x,\bar Q_T}^{(L)}$ are used. By $\mathcal{O}_{loc}^{n+L}((0,\infty)\times(0,\infty))$, we mean the set of all functions $q(\rho,\theta)^T$ which are defined on $(0,\infty)\times(0,\infty)$, n-times partially differentiable and their n-th order derivatives are locally Lipschitz continuous there.

2. Statement of the problem. It is natural and plausible, to the present author, that the movement of one fluid acts upon those of others and the movement necessarily accompanies heat change and vice versa, so that we consider the multi-phase free boundary problem arising from the movement of a finite number, say n, of nonmiscible general fluids.

Let Ω_0 [resp. $\Omega_1,\Omega_2,\cdots,\Omega_n$] be a bounded or unbounded domain in $\mathbf{R}^3$ [resp. Ω_0] with a boundary Γ_0 [resp. $\Gamma_1,\Gamma_2,\cdots,\Gamma_n$]; the distances between Γ_j and Γ_k $(j,k=0,1,\cdots,n;j\neq k)$ be supposed to be positive; the exterior boundary Γ_0 be assumed to be rigid. We set

$$\omega_0=\Omega_0-\bigcup_{i=1}^{n}\Omega_j,$$

$$\omega_j = \begin{cases} \Omega_j & \text{if there is no } k\in\{1,2,\cdots,n\}-\{j\} \text{ such that } \Omega_j\supset\Omega_k, \\ \Omega_j - \bigcup_{k:\Omega_k\subset\Omega_j} \Omega_k & \text{if there are } k\in\{1,2,\cdots,n\}-\{j\} \text{ such that } \Omega_j\supset\Omega_k. \end{cases}$$

Denoting by $\omega_j(t)$, the domain of the fluid at time t which initially occupies ω_j $(j=0,1,\cdots,n)$, then our problem consists of finding the domain $\omega_j(t)$ and the function $(\rho^{(j)},v^{(j)},\theta^{(j)})$ defined on $\omega_j(t)$ $(j=0,1,\cdots,n)$ satisfying the system of differential equations:

$$(1)\quad \begin{cases} [\frac{D}{Dt}]^{(j)}\rho^{(j)} = -\rho^{(j)}\nabla\cdot v^{(j)}, \\ \rho^{(j)}[\frac{D}{Dt}]^{(j)}v^{(j)} = \nabla\, P^{(j)} + \rho^{(j)}f^{(j)}, \\ \rho^{(j)}\theta^{(j)}[\frac{D}{Dt}]^{(j)}S^{(j)} = \nabla\cdot(\kappa^{(j)}\nabla\theta^{(j)})+\mu'^{(j)}(\nabla\cdot v^{(j)})^2+2\mu^{(j)}D^{(j)}:D^{(j)} \end{cases}$$

$$\text{in } \mathcal{O}_T^{(j)} \equiv \quad ,t)\in\mathbb{R}^4 \mid x\in\omega_j(t),\ t\in(0,T)\}\ (T>0),$$

the initial conditions

$$(2)\qquad (\rho^{(j)},v^{(j)},\theta^{(j)})|_{t=0} = (\rho_0^{(j)},v_0^{(j)},\theta_0^{(j)})(x) \quad (x\in\omega_j),$$

the boundary conditions

$$(3)\quad \begin{cases} v^{(j)} = v^{(j')}, \quad P^{(j)}n^{(j)}(x,t) = P^{(j')}n^{(j')}(x,t), \\ \theta^{(j)} = \theta^{(j')}, \quad \kappa^{(j)}\nabla\theta^{(j)}\cdot n^{(j)}(x,t) = \kappa^{(j')}\nabla\theta^{(j')}\cdot n^{(j')}(x,t) \end{cases}$$

$$\text{on } \partial\omega_{j,T}\cap\partial\omega_{j',T}$$

for $\forall j,j'\in\{0,1,\cdots,n\}$ $(j\neq j')$ satisfying $\partial\omega_{j,T}\cap\partial\omega_{j',T}\neq\emptyset$,

$$(4)\qquad v^{(0)} = 0 \text{ (non-slip condition)}, \quad \theta^{(0)} = \theta_e \quad \text{on } \Gamma_{0,T}\equiv\Gamma_0\times[0,T],$$

and the equations

$$(5)\qquad [\frac{D}{Dt}]^{(j=j')}F^{(j=j')}(x,t) = 0 \quad \text{on } \partial\omega_{j,T}\cap\partial\omega_{j',T}\ (\neq\emptyset).$$

Here $\rho^{(j)}=\rho^{(j)}(x,t)$ is the density, $v^{(j)}=v^{(j)}(x,t)=(v_1^{(j)},v_2^{(j)},v_3^{(j)})$ is the velocity of the fluid at time t at the point $x=(x_1,x_2,x_3)$, $f^{(j)}=f^{(j)}(x,t)$ is a vector of external forces and $\theta^{(j)}=\theta^{(j)}(x,t)$ is the absolute temperature. The pressure $p^{(j)}$, the entropy $S^{(j)}$, the coefficients of viscosity $\mu^{(j)}$ and $\mu'^{(j)}$, and the coefficient of heat conduction $\kappa^{(j)}$ are given functions of the variables $\rho^{(j)}$ and $\theta^{(j)}$ satisfying the conditions $\mu^{(j)}$, $\kappa^{(j)}$, $S^{(j)}_{\theta^{(j)}}(=\partial S^{(j)}/\partial\theta^{(j)}) > 0$, $2\mu^{(j)} + 3\mu'^{(j)} \geq 0$. $\nabla = (\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$; $[\frac{D}{Dt}]^{(j)} = \partial/\partial t + (v^{(j)}\cdot\nabla)$; $P^{(j)} = [-p^{(j)}+\mu'^{(j)}(\nabla\cdot v^{(j)})]I + 2\mu^{(j)}D^{(j)}$; I is the identity matrix of order 3; $D^{(j)}=D^{(j)}(v^{(j)})$ is a matrix with elements $D^{(j)}_{ik} = \frac{1}{2}(\partial v_i^{(j)}/\partial x_k + \partial v_k^{(j)}/\partial x_i)$, $i,k=1,2,3$; $D^{(j)}:D^{(j)}=\sum_{i,k=1}^{3} D_{ik}^{(j)}D_{ik}^{(j)}$; $\partial\omega_j(t)$ is the boundary of $\omega_j(t)$; $\partial\omega_{j,T}=\{(x,t)\,|\,x\in\partial\omega_j(t),\ t\in[0,T]\}$; $F^{(j=j')}(x,t)$ is such as $\partial\omega_j(t)\cap\partial\omega_{j'}(t) = \{x\in R^3 \mid F^{(j=j')}(x,t) = 0\}$; $n^{(j)}=n^{(j)}(x,t)$ is a unit normal vector at $x\in\partial\omega_j(t)\cap\partial\omega_{j'}(t)$ pointing into the interior of $\omega_j(t)$ $(n^{(j')}=-n^{(j)})$.

Throughout this paper we assume that the compatibility conditions are valid even if they are not written down explicitly.

Our main result is the following:

Theorem. *Suppose* (i) $\Gamma_0,\Gamma_1,\cdots,\Gamma_n\in C^{2+\alpha}(\alpha\in(0,1))$, $dis(\Gamma_j,\Gamma_k)>0$ $(j,k=0,1,\cdots,n;j\neq k)$ (ii) $(\rho_0^{(j)},v_0^{(j)},\theta_0^{(j)})\in C^{1+\alpha}(\bar{\omega}_j)\times C^{2+\alpha}(\bar{\omega}_j)\times C^{2+\alpha}(\bar{\omega}_j)$ $(0<\underline{\rho}_0^{(j)}\leq\rho_0^{(j)}(x)\leq\bar{\rho}_0^{(j)},\ 0<\underline{\theta}_0^{(j)}\leq\theta_0^{(j)}(x)\leq\bar{\theta}_0^{(j)};\ \underline{\rho}_0^{(j)},\ \bar{\rho}_0^{(j)},\ \underline{\theta}_0^{(j)},\ \bar{\theta}_0^{(j)}$ *are constants*) $(j=0,1,\cdots,n)$ (iii) $f^{(j)}\in B^1(\bar{R}_T^3\equiv R^3\times[0,T])$, $\sum_{r+|s|=1}|D_t^rD_x^sf^{(j)}|_{x,\bar{R}_T^3}^{(L)}<+\infty$ $(j=0,1,\cdots,n)$ (iv) $\mu^{(j)},\ \mu'^{(j)},\ \kappa^{(j)},\ p^{(j)},\ S^{(j)}\in\mathcal{C}_{loc}^{2+L}((0,\infty)\times(0,\infty))$, $2\mu^{(j)}+3\mu'^{(j)}\geq 0$, $\mu^{(j)},\ \kappa^{(j)},\ S_{\theta^{(j)}}^{(j)}>0$ $(j=0,1,\cdots,n)$ (v) $\theta_e\in C_{x,t}^{2+\alpha,1+\alpha/2}(\Gamma_{0,T})$.

Then there exists a unique solution $(\rho^{(j)},v^{(j)},\theta^{(j)})$ $(j=0,1,\cdots,n)$ *of* (1)~(5), *which belongs to* $B^{1+\alpha}(\bar{\mathcal{Q}}_{T'}^{(j)})\times C_{x,t}^{2+\alpha,1+\alpha/2}(\bar{\mathcal{Q}}_{T'}^{(j)})\times C_{x,t}^{2+\alpha,1+\alpha/2}(\bar{\mathcal{Q}}_{T'}^{(j)})$ $(0<o^{(j)}\leq\bar{\rho}^{(j)}=$*constant*, $0<\theta^{(j)}\leq\bar{\theta}^{(j)}=$*constant*) *for some* $T'\in(0,T)$ $(j=0,1,\cdots,n)$.

3. Sketch of the proof of Theorem.

Since we have already proved in [8] the analogous theorem for two-phase free boundary problem of general fluids in detail and the same arguements are applicable in the present case, we give here only the sketch of the proof of the above theorem.

1°. First of all, we transform the equations (1) by the characteristic transformation $\Pi_{x_0,t_0}^{x,t}:(x,t)\mapsto(x_0,t_0)$ which is defined by the relation

$$x=x_0+\int_0^{t_0}\hat{v}^{(j)}(x_0,\tau)d\tau\equiv x(x_0,t_0;\hat{v}^{(j)})\quad(\hat{v}^{(j)}(x_0,t_0)=\Pi_{x_0,t_0}^{x,t}v^{(j)}(x,t))$$

into the form

$$(6)\quad\begin{cases}\dfrac{\partial}{\partial t_0}\hat{\rho}^{(j)}=-\hat{\rho}^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)},\\ \hat{\rho}^{(j)}\dfrac{\partial}{\partial t_0}\hat{v}^{(j)}=\nabla_{\hat{v}^{(j)}}(\mu'^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)})+2\nabla_{\hat{v}^{(j)}}\cdot(\mu D_{\hat{v}^{(j)}}(\hat{v}^{(j)}))-\\ \qquad-\nabla_{\hat{v}^{(j)}}p^{(j)}+\hat{\rho}^{(j)}\hat{f}^{(j)},\\ \hat{\rho}^{(j)}\theta^{(j)}S_{\hat{\theta}^{(j)}}^{(j)}\dfrac{\partial}{\partial t_0}\theta^{(j)}=\nabla_{\hat{v}^{(j)}}\cdot(\kappa^{(j)}\nabla_{\hat{v}^{(j)}}\theta^{(j)})+\mu'^{(j)}(\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)})^2+\\ \qquad+2\mu^{(j)}D_{\hat{v}^{(j)}}(\hat{v}^{(j)}):D_{\hat{v}^{(j)}}(\hat{v}^{(j)})+\hat{\rho}^{(j)2}\theta^{(j)}S_{\hat{\rho}^{(j)}}^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)}\end{cases}$$

$(j=0,1,\cdots,n)$.

Here $(\hat{\rho}^{(j)},\hat{\theta}^{(j)})(x_0,t_0)=\Pi_{x_0,}^{x,t}(\rho^{(j)},\theta^{(j)})(x,t)$, $\mathcal{Q}^{(j)}=(\partial x(x_0,t_0,v^{(j)})/\partial x_0)^{-1}$,

$\nabla_{\hat{v}^{(j)}} = (\nabla_{\hat{v}^{(j)},1}, \nabla_{\hat{v}^{(j)},2}, \nabla_{\hat{v}^{(j)},3}) = \mathcal{G}^{(j)}\overset{\circ}{\nabla}$, $\overset{\circ}{\nabla} = (\partial/\partial x_{0,1}, \partial/\partial x_{0,2}, \partial/\partial x_{0,3})$,

$D_{\hat{v}^{(j)}}(\hat{v}^{(j)})$ is a matrix with elements $\frac{1}{2}(\nabla_{\hat{v}^{(j)},k}\hat{v}_i^{(j)} + \nabla_{\hat{v}^{(j)},i}\hat{v}_k^{(j)})$ (i,k=1,2,3).

Integrating the equations $(6)_1$, we can reduce our problem to the initial-boundary value problem for the parabolic system $(6)_{2,3}$ with

$$\hat{\rho}^{(j)}(x_0,t_0) = \rho_0^{(j)}(x_0)\exp[-\int_0^{t_0}\nabla_{\hat{v}^{(j)}}\hat{v}^{(j)}(x_0,\tau)d\tau]$$

and with the initial-boundary conditions

(7) $$(\hat{v}^{(j)}, \hat{\theta}^{(j)})(x_0,0) = (v_0^{(j)}, \theta_0^{(j)})(x_0) \quad \text{on } \omega_j \ (j=1,2,\cdots,n),$$

(8) $$\begin{cases} \hat{v}^{(j)} = \hat{v}^{(j')}, \quad \dfrac{\hat{P}^{(j)}\cdot\mathcal{G}^{(j)}n^{(j)}(x_0)}{|\mathcal{G}^{(j)}\overset{\circ}{\nabla}F_0^{(j)}|} = \dfrac{\hat{P}^{(j')}\cdot\mathcal{G}^{(j')}n^{(j')}(x_0)}{|\mathcal{G}^{(j')}\overset{\circ}{\nabla}F_0^{(j')}|}, \quad \hat{\theta}^{(j)} = \hat{\theta}^{(j')}, \\ \dfrac{\kappa^{(j)}\mathcal{G}^{(j)}n^{(j)}(x_0)}{|\mathcal{G}^{(j)}\overset{\circ}{\nabla}F_0^{(j)}|}\cdot(\nabla_{\hat{v}^{(j)}}\hat{\theta}^{(j)}) = \dfrac{\kappa^{(j')}\mathcal{G}^{(j')}n^{(j')}(x_0)}{|\mathcal{G}^{(j')}\overset{\circ}{\nabla}F_0^{(j')}|}\cdot(\nabla_{\hat{v}^{(j')}}\hat{\theta}^{(j')}) \end{cases}$$

on $[\partial\omega_j \cap \partial\omega_{j'}]\times[0,T]$ for $\forall j,j' \in \{0,1,\cdots,n\}$ $(j\neq j')$ satisfying $\partial\omega_j \cap \partial\omega_{j'} \neq \emptyset$,

(9) $$\hat{v}^{(0)} = 0, \quad \hat{\theta}^{(0)} = \hat{\theta}_e \quad \text{on } \Gamma_{0,T},$$

where $\hat{P}^{(j)} = [-p^{(j)} + \mu'^{(j)}\nabla_{\hat{v}^{(j)}}\cdot\hat{v}^{(j)}]I + 2\mu^{(j)}D_{\hat{v}^{(j)}}(\hat{v}^{(j)})$, $F_0^{(j)}(x_0) = F^{(j)}(x_0,0)$, $n^{(j)}(x_0) = n^{(j)}(x_0,0)$.

(6)∿(9) can be written in a shorter form

(10) $$\begin{cases} \dfrac{\partial}{\partial t_0}w^{(j)} = \mathcal{A}^{(j)}(x_0,t_0,w^{(j)};\overset{\circ}{\nabla})w^{(j)} + \mathcal{B}^{(j)}(x_0,t_0,w^{(j)}) \quad \text{in } Q_T^{(j)}, \\ w^{(j)}|_{t_0=0} = 0, \\ w^{(0)} = (0, \hat{\theta}_e - \theta_0^{(0)}) \quad \text{on } \Gamma_{0,T}, \\ \begin{pmatrix} I \\ \dfrac{B^{(j)}(x_0,t_0,w^{(j)};\overset{\circ}{\nabla})}{|\mathcal{G}^{(j)}n^{(j)}(x_0)|} \end{pmatrix} w^{(j)} - \begin{pmatrix} I \\ \dfrac{B^{(j')}(x_0,t_0,w^{(j')};\overset{\circ}{\nabla})}{|\mathcal{G}^{(j')}n^{(j')}(x_0)|} \end{pmatrix} w^{(j')} = \\ \qquad = \phi(x_0,t_0,w^{(j)},w^{(j')}) \quad \text{on } [\partial\omega_j \cap \partial\omega_{j'}]\times[0,T], \end{cases}$$

where $w^{(j)} = (\hat{v}^{(j)} - v_0^{(j)}, \hat{\theta}^{(j)} - \theta_0^{(j)})$, $Q_T^{(j)} = \omega_j\times(0,T)$, $\mathcal{A}^{(j)}(x_0,t_0,w^{(j)};\overset{\circ}{\nabla})$ and $B^{(j)}(x_0,t_0,w^{(j)};\overset{\circ}{\nabla})$ are matrices with elements 2nd and 1st order differential operators respectively.

2°. We consider a linearlized initial-boundary value problem of (10):

$$
(11)\quad \begin{cases}
\dfrac{\partial}{\partial t_0} w^{(j)} = \mathcal{A}^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})w^{(j)} + \mathcal{B}^{(j)}(x_0,t_0,w^{(j)}) \quad \text{in } Q_T^{(j)},\\
w^{(j)}|_{t_0=0} = 0,\\
w^{(0)} = (0,\hat{\vartheta}_e - \vartheta_0^{(0)}) \quad \text{on } \Gamma_{0,T},\\
\left(\dfrac{B^{(j)}(x_0,t_0,w^{(j)};\mathring{\nabla})}{|\mathcal{G}^{(j)}(w^{(j)})n^{(j)}(x_0)|}\right)^{I} w^{(j)} - \left(\dfrac{B^{(j')}(x_0,t_0,w^{(j')};\mathring{\nabla})}{\mathcal{G}^{(j')}(w^{(j')})n^{(j')}(x_0)}\right)^{I} w^{(j')} =\\
= \phi(x_0,t_0,w^{(j)},w^{(j')}) \quad \text{on } [\partial\omega_j \cap \partial\omega_{j'}] \times [0,T].
\end{cases}
$$

Here $w^{(j)}(j=0,1,\cdots,n)$ are assumed to belong to the set

$$
\mathfrak{S}_T = \{(w^{(0)},\cdots,w^{(n)}) \in C^{2+\alpha,1+\alpha/2}_{x_0,t_0}(\bar{Q}_T^{(0)}) \times \cdots \times C^{2+\alpha,1+\alpha/2}_{x_0,t_0}(\bar{Q}_T^{(n)}) \mid w^{(j)}|_{t_0=0} = 0,
$$
$$
\|w^{(j)}\|^{(2)}_{\bar{Q}_T^{(j)}} < M_1^{(j)}, \quad |\mathring{\nabla}\mathring{\nabla} w^{(j)}|^{(\alpha)}_{x_0,\bar{Q}_T^{(j)}} < M_2^{(j)} \quad (j=0,1,\cdots,n)\}
$$

$\left(\|w^{(j)}\|^{(2)}_{\bar{Q}_T^{(j)}} = \sum_{2r+|s|=0}^{2} |D_{t_0}^r D_{x_0}^s w^{(j)}|^{(0)}_{\bar{Q}_T^{(j)}}\right)$ for any positive number $M_1^{(j)}$ and a positive number $M_2^{(j)}$ determined later.

We note the two facts:

(a) The system of differential equations (11) is uniformly parabolic in the sense of Petrowsky (modulo of parabolicity δ) for a suitably chosen T.

(b) When we consider the same problem as (11) in $\mathbb{R}^3_+ \equiv \{x_0=(x_{0,1},x_{0,2},x_{0,3}) \mid x_{0,3} > 0\}$, the complementing condition holds (see [6,8]).

(b) garantees the possibility for the construction of the regularizer of (11) in the half space $\mathbb{R}^3_+$, from which, together with the partition of unity, follows the solvability of auxiliary linearized problem (11):

There exists a unique solution $w^{(j)} \in C^{2+\alpha,1+\alpha/2}_{x_0,t_0}(\bar{Q}_T^{(j)})$ of (11) satisfying the estimates

$$
(12)\quad \begin{cases}
\|w^{(j)}\|^{(2)}_{\bar{Q}_T^{(j)}} \le [c_1^{(j)}(T,M_1^{(j)}) + c_2^{(j)}(T,M_1^{(j)})M_2^{(j)}](T^{\alpha/2} + T^{1+\alpha/2}),\\
|\nabla\nabla w^{(j)}|^{(\alpha)}_{x_0,\bar{Q}_T^{(j)}} \le c_1^{(j)}(T,M_1^{(j)}) + c_2^{(j)}(T,M_1^{(j)})M_2^{(j)},
\end{cases}
$$

where $c_1^{(j)}$ and $c_2^{(j)}$ increase monotonically in T and $M_1^{(j)}$ and $c_2^{(j)} \to 0$ as $T \to 0$ $(j=0,1,\cdots,n)$. If we choose the constant $M_2^{(j)}$ and T_0 in such a way that $M_2^{(j)} > c_1^{(j)}(T,M_1^{(j)}) + M$ for any positive number M and for such $M_2^{(j)}$, $[c_1^{(j)}(T_0,M_1^{(j)}) + M]\ (T_0^{\alpha/2} + T_0^{1+\alpha/2}) \le M_1^{(j)}$ and $c_2^{(j)}(T_0,M_1^{(j)})M_2^{(j)} \le M$, then $W = (W^{(0)}, \cdots, W^{(n)}) \in \mathfrak{S}_{T_0}$. We denote T_0 by T for simplicity.

3°. Next we construct the sequence $\{w_m(x_0,t_0)\}$ of successive approximate solutions as follows:

$$\begin{cases} w_0(x_0,t_0) = 0 \\ w_m(x_0,t_0) \text{ is defined as a solution } W^{T'} \text{ of (11) assuming } w \equiv (w^{(0)}, \cdots, \\ w^{(n)}) = w_{m-1} \in \mathfrak{S}_T. \end{cases}$$

Then the result in 2°implies that w_m $(m=0,1,2,\cdots)$ are well defined and belong to $\mathfrak{S}_T$. Applying the estimates (12) to the equation concerning $w_m - w_{m-1}$, we obtain

$$|||w_m - w_{m-1}||| \leq C_3(T,M_1,M_2)\,|||w_{m-1} - w_{m-2}||| \tag{13}$$

$(|||w||| \equiv \sum_{j=0}^{n} \|w^{(j)}\|_{\bar{Q}_T^{(j)}}^{(2+\alpha)}$, $M_1 = \sum_{j=0}^{n} M_1^{(j)}$, $M_2 = \sum_{j=0}^{n} M_2^{(j)})$ where $C_3 \to 0$ as $T \to 0$.

Therefore the sequence $\{w_m(x_0,t_0)\}$ converges to $w(x_0,t_0)$ uniformly if we choose $T' \in (0,T]$ so as to satisfy $C_3(T',M_1,M_2) < 1$. Then $\hat{v} \equiv (v^{(0)}, \cdots, v^{(n)}) = w' + v_0$ $(w = (w',w_4)$, $v_0 = (v_0^{(0)}, \cdots, v_0^{(n)}))$, $\hat{\theta} = w_4 + \theta_0$ $(\theta_0 = (\theta_0^{(0)}, \cdots, \theta_0^{(n)}))$, $\hat{\rho}(x_0,t_0) = \rho_0(x_0) \exp[-\int_0^{t_0} \hat{v}\, d\tau]$ is our desired solution of (6)∿(9). The uniqueness of the solution follows from the uniqueness of the solution of (10), which is proved by the fact that two solutions supposed to exist satisfy the inequality analogous to (13).

4°. The unique solution of the original free boundary problem (1)∿(5) can be obtained by the formulae

$$(\rho^{(j)}(x,t), v^{(j)}(x,t), \theta^{(j)}(x,t), \omega_j(t)) = \Pi_{x,t}^{x_0,t_0}(\hat{\rho}^{(j)}(x_0,t_0), \hat{v}^{(j)}(x_0,t_0), \hat{\theta}^{(j)}(x_0,t_0), \omega_j) \quad (j=0,1,\cdots,n).$$

The positivity and boundedness of ρ and θ are obvious from our construction method.

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A GENERALIZATION OF THE THEOREM OF MAULDIN

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Abstract: For a perfect Polish space X and a σ-ideal $\mathcal{J}$ of subsets of X, let $\Phi(X,\mathcal{J})$ denote the family of all real-valued functions on X continuous almost everywhere with respect to $\mathcal{J}$. We shall prove that the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for a general class of σ-ideals $\mathcal{J}$, thus generalizing the Mauldin's result for X = [0,1] and the sets of Lebesgue measure zero for $\mathcal{J}$.

Key words: Baire classes of functions, σ-ideals of sets.

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Let X be a perfect Polish space. We consider σ-ideals of subsets of X. It is assumed that each σ-ideal contains all singletons $\{x\}$ and does not contain any nonempty open subset of X. For a fixed σ-ideal $\mathcal{J}$, let $\Phi(X,\mathcal{J})$ denote the family of all real-valued functions defined on X which are continuous almost everywhere with respect to $\mathcal{J}$. Suppose that a σ-ideal $\mathcal{J}_0$ is such that the following conditions hold:

(I) there is a compact subset X_0 of X which does not belong to $\mathcal{J}_0$;

(II) for each countable subset A of X, there is a G_δ-set belonging to $\mathcal{J}_0$ such that $A \subseteq B$.

It is proved that the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_0$. Mauldin [8] obtained this result in the case when X is the unit interval and $\mathcal{J} = \mathcal{J}_0$ is the σ-ideal of all sets of the Lebesgue measure zero. Our proof is based on the method presented in [8]. We also use topological properties concerning σ-ideals (for instance, a generalization of the Cantor-Bendixson Theorem is proved). The main result of this note can be applied to the σ-ideal constructed by Mycielski in [10].

Let X be a set and let Φ be a family of real-valued functions defined on X. We define $\Phi_0 = \Phi$ and, for each ordinal $\alpha > 0$, let Φ be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma<\alpha} \Phi_\gamma$. The first uncountable ordinal will be denoted by ω_1. Observe that $\Phi_{\omega_1} = \Phi_{\omega_1+1}$ and Φ_{ω_1} is the smallest subfamily of $\mathbb{R}^X$ which contains Φ and which is closed with respect to pointwise limits of sequences. The Baire order of Φ is a first ordinal α such that $\Phi_\alpha = \Phi_{\alpha+1}$. For example, if Φ denotes the family of all real-valued functions defined on the unit interval, then the Baire order of Φ is ω_1 [11].

Now, let X be a perfect Polish space. Consider those σ-ideals of subsets of X which contain all singletons $\{x\}$ and do not contain any nonempty open subset of X. For a fixed σ-ideal $\mathcal{J}$, let $\Phi = \Phi(X,\mathcal{J})$ be the family of all real-valued functions on X whose set of points of discontinuity belongs to $\mathcal{J}$. Notice that the Baire order of $\Phi(X,\mathcal{J})$ is always positive because the characteristic function of any countable dense subset of X belongs to $\Phi_1(X,\mathcal{J}) \setminus \Phi_0(X,\mathcal{J})$ (we write $\Phi_\alpha(X,\mathcal{J})$ instead of $(\Phi(X,\mathcal{J}))_\alpha$). The problems connected with the Baire order of $\Phi(X,\mathcal{J})$ were studied by Mauldin in [6],[7],[8],[9]. It is known that the order of $\Phi(X,\mathcal{J})$ equals 1 if $\mathcal{J}$ denotes the σ-ideal of all sets of the first category [2]. Mauldin in [8] proved that if X is the unit interval and $\mathcal{J}$ denotes the σ-ideal of all sets of the Lebesgue measure zero, then the order of $\Phi(X,\mathcal{J})$

is ω_1. Several generalizations of this result were obtained in [7]. Another generalization will be presented in this paper.

Mauldin in [6] gave the following characterization of the generalized Baire classes:

Theorem 1. If α is an ordinal, $0<\alpha<\omega_1$, then a function f is in $\Phi(X,\mathcal{J})$ if and only if there is a function g in the Baire class α such that the set $\{x: f(x) \neq g(x)\}$ is a subset of an F_σ set belonging to $\mathcal{J}$.

The Baire order of $\Phi(X,\mathcal{J})$ treated as a function of $\mathcal{J}$ is monotonic in the following sense:

Proposition 1. If $\mathcal{J}$ and $\mathcal{Y}$ are σ-ideals of subsets of X and $\mathcal{J} \subseteq \mathcal{Y}$, then the order of $\Phi(X,\mathcal{Y})$ is not greater than the order of $\Phi(X,\mathcal{J})$.

Proof. Let α be the order of $\Phi(X,\mathcal{J})$. Observe that it is enough to demonstrate the inclusion

$$\Phi_{\alpha+1}(X,\mathcal{Y}) \subseteq \Phi_{\alpha}(X,\mathcal{Y}).$$

It obviously holds if $\alpha = \omega_1$. Let $\alpha < \omega_1$. If f belongs to $\Phi_{\alpha+1}(X,\mathcal{Y})$, then, by Theorem 1, there exists a function g in the Baire class $\alpha + 1$ such that the set $\{x: f(x) \neq g(x)\}$ is a subset of an F_σ set belonging to $\mathcal{Y}$. Of course, g belongs to $\Phi_{\alpha+1}(X,\mathcal{J})$. Then, from the definition of α it follows that g belongs to $\Phi_{\alpha}(X,\mathcal{J})$. Hence, by Theorem 1, there exists a function h in the Baire class α such that the set $\{x: g(x) \neq h(x)\}$ is a subset of an F_σ set belonging to $\mathcal{J}$. Since $\mathcal{J} \subseteq \mathcal{Y}$, the set $\{x: f(x) \neq h(x)\}$ is a subset of an F_σ set belonging to $\mathcal{Y}$. Hence, by Theorem 1, the function f belongs to $\Phi_{\alpha}(X,\mathcal{Y})$.

The main result of this note is:

Theorem 2. Let $\mathcal{J}_0$ be a σ-ideal of subsets of X such that the following conditions hold:

(I) there is a compact subset X_0 of X which does not belong to $\mathcal{J}_0$;

(II) for each countable subset A of X, there is a G_δ set B belonging to $\mathcal{J}_0$ such that $A \subseteq B$.

Then the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_0$.

Remark. Considering X equal to the unit interval and $\mathcal{J}$, $\mathcal{J}_0$ equal to the σ-ideal of sets of the Lebesgue measure zero, we get the theorem of Mauldin [8].

In virtue of Proposition 1, we shall prove Theorem 2 if we only verify that the order of $\Phi(X,\mathcal{J}_0)$ is ω_1. The argument of this fact will be based on the method presented in [8].

The proof of Mauldin begins with a construction of a family which consists of perfect sets A such that if an open set V intersects A, then the set $V \cap A$ has positive measure. We shall generalize that property.

Let $\mathcal{J}$ be a σ-ideal of subsets of X.

Definition 1 (compare [4]). A closed nonempty subset A of X will be called $\mathcal{J}$-perfect if and only if, for each open set V such that V intersects A, we have $V \cap A \notin \mathcal{J}$.

Remark. Since $\mathcal{J}$ does not contain any nonempty open subset of X, the set X is $\mathcal{J}$-perfect.

Definition 2 (compare [10]). If A is a subset of X, then let $A^{(\mathcal{J})}$ denote the set of all points x of X such that, for each neighbourhood V of x, we have $V \cap A \notin \mathcal{J}$.

Let us quote from [10] a few properties of the operation

$A^{(\mathcal{J})}$:

(i) $A^{(\mathcal{J})}$ is closed and included in the closure of A;

(ii) $(A^{(\mathcal{J})})^{(\mathcal{J})} = A^{(\mathcal{J})}$;

(iii) $A \setminus A^{(\mathcal{J})} \in \mathcal{J}$.

Proposition 2. A nonempty subset A of X is $\mathcal{J}$-perfect if and only if $A = A^{(\mathcal{J})}$.

Proof. Assume that A is $\mathcal{J}$-perfect. Then, immediately from the definitions it follows that $A \subseteq A^{(\mathcal{J})}$. Since A is closed, therefore, by (i), we have $A^{(\mathcal{J})} \subseteq A$. Conversely, assume that $A = A^{(\mathcal{J})}$. Then, by (i), the set A is closed. Let an open set V intersect A. Consider a point which belongs to $V \cap A$. Then it belongs to A and from Definition 2 it follows that $V \cap A \notin \mathcal{J}$. Thus A is $\mathcal{J}$-perfect.

Proposition 3. For each closed subset A of X, there is a unique decomposition $A = B \cup C$ into disjoint sets such that B is empty or $\mathcal{J}$-perfect, and $C \in \mathcal{J}$.

Proof. If $A \in \mathcal{J}$, then we put $B = \emptyset$, $C = A$, and $A = B \cup C$ is the required unique decomposition. If $A \notin \mathcal{J}$, then we put $B = A^{(\mathcal{J})}$, $C = A \setminus B$. In virtue of (iii), we have $C \in \mathcal{J}$. Since $A \notin \mathcal{J}$, therefore $B \notin \mathcal{J}$. Hence B is nonempty and it follows from (ii) that $B^{(\mathcal{J})} = B$. Thus, in virtue of Proposition 2, the set B is $\mathcal{J}$-perfect. Now, assume that $A = B' \cup C'$ where B', C' are disjoint, B' is $\mathcal{J}$-perfect and $C' \in \mathcal{J}$. If $x \in B'$ and V is any neighbourhood of x, then $V \cap B'$ $\mathcal{J}$. Hence $V \cap A \notin \mathcal{J}$ and $x \in A^{(\mathcal{J})}$. Thus $B' \subseteq B$. If $x \in C'$, then there is a neighbou hood V of x such that $V \cap B' = \emptyset$ since B', C' are disjoint and B' is closed. Now, $V \cap B' = \emptyset$ implies $V \cap A = V \cap C'$ and then $V \cap A \in \mathcal{J}$.

Hence $x \in C$. So, we have $B' \subseteq B$, $C' \subseteq C$. Since $B \cup C = B' \cup C'$ and $B \cap C = \emptyset = B' \cap C'$, there must be $B = B'$, $C = C'$.

Remarks. Martin in [5] explored topologies generated by the operation of the derived set. Notice that $A^{(\mathcal{J})}$ is such an operation. Then $A \cup A^{(\mathcal{J})}$ is a closure operation and it generates a topology which we denote by $\mathcal{T}'$ (comp. [1],[5],[10]). From [5], Th. 1, it follows that if $x \in A^{(\mathcal{J})}$ implies $x \in (A \setminus \{x\})^{(\mathcal{J})}$, then the derived set of A in the topology $\mathcal{T}'$ coincides with $A^{(\mathcal{J})}$. We have assumed that $\{x\} \in \mathcal{J}$ for each $x \in X$, therefore the above-mentioned condition holds. Thus, Proposition 2 means that $\mathcal{J}$-perfect sets are identical with perfect sets in the topology $\mathcal{T}'$. Proposition 3 is a kind of generalization of the Cantor-Bendixson Theorem. Similar results were obtained in [1] (Satz II) and [4](Th. 1.3).

Now, suppose that $\mathcal{J}_0$ and X_0 fulfil all the hypotheses of Theorem 2. Since X_0 is closed and $X_0 \notin \mathcal{J}_0$, therefore by Proposition 3, there is an $\mathcal{J}_0$-perfect set $X_* \subseteq X_0$. Of course, X_* is compact. Let

$$\mathcal{J}_0^* = \{A \cap X_* : A \in \mathcal{J}_0\}.$$

Observe that $\mathcal{J}_0^* \subseteq \mathcal{J}_0$ and $\mathcal{J}_0^*$ is a σ-ideal of subsets of the perfect Polish space X_* .

Lemma 1 (compare [9], Th. 2). The Baire order of $\Phi(X_*, \mathcal{J}_0^*)$ is not greater than the Baire order of $\Phi(X, \mathcal{J}_0)$.

Proof. Suppose that the order of $\Phi(X_*, \mathcal{J}_0^*)$ is greater than the order of $\Phi(X, \mathcal{J}_0)$. Thus, the order of $\Phi(X, \mathcal{J}_0)$ equals a countable ordinal α . Let f belong to $\Phi_{\alpha+1}(X_*, \mathcal{J}_0^*)$. Then, by Theorem 1, there is a function g defined on X_* which is in the Baire class $\alpha + 1$, such that the set

$$A = \{x: f(x) \neq g(x)\}$$

is a subset of a set B which is of type F_σ with respect to X_* and belongs to $\mathcal{J}_0^*$. Let $\hat{f}$, $\hat{g}$ be extensions of f, g, respectively, to the whole X, such that $\hat{f}(x) = \hat{g}(x) = 0$ for $x \in X \setminus X^*$. Then $\hat{g}$ belongs to the Baire class $\alpha + 1$ and we have $\{x: \hat{f}(x) \neq \hat{g}(x)\} = A$. As above, $A \subseteq B$ and one can easily check that B is an F_σ set with respect to X, belonging to $\mathcal{J}_0$. Thus, by Theorem 1, $\hat{f}$ belongs to $\Phi_{\alpha+1}(X, \mathcal{J}_0)$. Hence $\hat{f}$ is in $\Phi_\alpha(X, \mathcal{J}_0)$ by the definition of α. It can be shown by transfinite induction that, for all γ, $0 \leq \gamma < \omega_1$, if a function is in $\Phi_\gamma(X, \mathcal{J}_0)$, then its restriction to X_* is in $\Phi_\gamma(X_*, \mathcal{J}_0^*)$. Therefore the function f, which is the restriction of $\hat{f}$ to X_*, belongs to $\Phi_\alpha(X_*, \mathcal{J}_0^*)$. So, it follows that $\Phi_\alpha(X_*, \mathcal{J}_0^*) = \Phi_{\alpha+1}(X_*, \mathcal{J}_0^*)$. This contradicts the assumption that the order of $\Phi(X_*, \mathcal{J}_0^*)$ is greater than α.

Now, in virtue of Lemma 1, it is enough to prove that the Baire order of $\Phi(X_*, \mathcal{J}_0^*)$ equals ω_1. Thus, we shall consider X_*, $\mathcal{J}_0^*$ instead of X, $\mathcal{J}_0$, respectively. For simplicity, we shall preserve the notation X, $\mathcal{J}_0$. We shall only add the assumption that X is compact. Observe that the condition (II) is still true.

Lemma 2. For each F_σ subset D of X such that $D \notin \mathcal{J}_0$ there is a set D_0 included in D such that D_0 is $\mathcal{J}_0$-perfect and nowhere dense in D.

Proof. Let A be a countable subset of D, dense in D. Since the condition (II) holds, there is a G_δ set $B \in \mathcal{J}_0$ such that $A \subseteq B$. Let $E = D \setminus B$. The set E is of type F_σ, of the first category in D, and $E \notin \mathcal{J}_0$. Let $E = \bigcup_{n=1}^{\infty} E_n$ where E_n are closed and nowhere dense in D. Then there exists $E_{n_0} \notin \mathcal{J}_0$. In virtue of

Proposition 3, there exists a set D_o which is contained in E_{n_o} and $\mathcal{J}_o$-perfect. The set D_o just fulfils the conclusion.

Lemma 3. For each $\mathcal{J}_o$-perfect set P, for each nonempty set V open with respect to P, and for each closed set F_o contained in P and nowhere dense in P, there is a set D_o included in $V \setminus F_o$ which is $\mathcal{J}_o$-perfect and nowhere dense in P.

Proof. It is enough to apply Lemma 2 to the set $D = V \setminus F_o$.

The following lemma can be proved by using Lemma 3 and repeating Mauldin's construction (see [8], the proof of Lemma 1).

Lemma 4. Let P be an $\mathcal{J}_o$-perfect set. There is a double sequence $\{F_{nk}\}_{n,k=1}^{\infty}$ of disjoint subsets of P such that

(a) each F_{nk} is $\mathcal{J}_o$-perfect and nowhere dense in P;

(b) if n is a natural number and V is a nonempty set open with respect to P, then there is some k such that F_{nk} is a subset of V.

The next part of the proof of Theorem 2 is analogous to that of [8]. Instead of the unit interval one considers the space X; moreover, the notations $\lambda(A) = 0$, $\lambda(A) > 0$ are to be replaced by $A \in \mathcal{J}_o$, $A \notin \mathcal{J}_o$, respectively (here $\lambda(A)$ means the Lebesgue measure of A).

In such a way we obtain the following lemma (compare [8], Lemma 4):

Lemma 5. There is an $F_{\sigma\delta}$ set H included in X and a Borel measurable function f from H onto the set $\mathcal{N}$ of all irrational numbers between 0 and 1, such that if $z \in \mathcal{N}$, then $f^{-1}(\{z\})$ is not a subset of an F_σ set belonging to $\mathcal{J}_o$.

The further two theorems play the same role as Theorems 1 and 2 in [8].

The countable product of identical sets which are all equal to X will be denoted by X^{ω_0}. Assume that X^{ω_0} is equipped with the Tychonoff topology. Notice that X^{ω_0} forms a Polish space.

Theorem 3. There is a Borel measurable mapping h from X onto X^{ω_0} such that if $t \in X^{\omega_0}$, then $h^{-1}(\{t\})$ is not a subset of an F_σ set belonging to $\mathcal{J}_0$.

Proof. Let f be a function described in Lemma 5. Since X^{ω_0} is a Polish space, there exists a continuous mapping g of $\mathcal{N}$ onto X^{ω_0} (see [3], p. 353, Th. 1). Consider $x_0 \in X$ and put

$$h(x) = \begin{cases} g(f(x)) & \text{if } x \in H \\ (x_0, x_0, x_0, \ldots) & \text{if } x \in X \setminus H. \end{cases}$$

The mapping h has the required properties.

Theorem 4. There exists a transfinite sequence of "universal functions" $\{U_\alpha\}_{0<\alpha<\omega_1}$ such that, for each α, $0 < \alpha < \omega_1$, we have

(1) U_α is a Borel measurable function on $X \times X$ into the unit interval I,

(2) if f is a function in the Baire class α, which maps X into I, then the set of all x, such that $U_\alpha(x,y) = f(y)$ for each y in X, is not a subset of an F_σ set belonging to $\mathcal{J}_0$.

Proof (cf. [11], p.339). Since X is compact and I is separable, then the space of all continuous functions on X into I with the topology generated by the uniform convergence is separable (see [3], p.120, Th.2). Let $\{S_n\}_{n=1}$ be a countable dense subset of this space. Choose an arbitrary sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points of X. For $(x,y)\in X\times X$, let

$$U_0(x,y) = \begin{cases} S_n(y) & \text{if } x = x_n \\ 0 & \text{otherwise.} \end{cases}$$

Let $h = (h_1, h_2, h_3, \ldots)$ be a mapping described in Theorem 3. For each ordinal α, $0 \leq \alpha < \omega_1$, and for each $(x,y)\in X\times X$, let

$$U_{\alpha+1}(x,y) = \limsup_{n\to\infty} U_\alpha(h_n(x),y).$$

If α is a limit ordinal, then let $\{\gamma_n\}_{n=1}^{\infty}$ be an increasing sequence of ordinals less than α which converges to α, and let

$$U_\alpha(x,y) = \limsup_{n\to\infty} U_{\gamma_n}(h_n(x),y).$$

Using transfinite induction, one shows that the sequence $\{U_\alpha\}_{0<\alpha<\omega_1}$ has properties (1),(2) (see [8], the proof of Th.2).

Now, the last part of the proof of Theorem 2 can be given. Suppose that the order of $\Phi(X, \mathcal{J}_0)$ is $\alpha < \omega_1$. Let U_α be defined as above and let

$$f(x) = \lim_{n\to\infty} (1 - U_\alpha(x,x))^n, \quad x\in X.$$

Since $0 \leq U_\alpha(x,x) \leq 1$, the equation $f(x) = U_\alpha(x,x)$ never holds. By Theorem 4, (1), the function f is Borel measurable. So, f belongs to $\Phi_\alpha(X, \mathcal{J}_0)$. In virtue of Theorem 1, there is a function g in the Baire class α such that the set A of all x for which $f(x) \neq g(x)$ is a subset of an F_σ set belonging to $\mathcal{J}_0$. In virtue of Theorem 4, (2), the set B of all x, such that $U_\alpha(x,y) = g(y)$ for each y in X, is not a subset of an F_σ set belonging to $\mathcal{J}_0$. Hence there is a point x_0 which belongs to $B\setminus A$. Then we have $U_\alpha(x_0,y) = g(y)$ for each y in X, and $f(x_0) = g(x_0)$. In particular, for $y = x_0$, we obtain $f(x_0) = U_\alpha(x_0,x_0)$. This is a contradiction. The proof of Theorem 2 has been comp-

Example. Consider $X = \{0,1\}^{\omega_0}$ and assume that $\{0,1\}$, X are equipped with the discrete and the Tychonoff topologies, respectively. The space X is homeomorphic to the Cantor set and so, X is a compact and perfect Polish space. Mycielski in [10] defined a σ-ideal $\mathcal{J}_0$ of subsets of X such that the condition (II) is fulfilled. Since X is compact, the condition (I) also holds. Hence, by Theorem 2, the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_0$. Let ν be a measure on $\{0,1\}$ such that $\nu(\{0\}) = \nu(\{1\}) = {}^1/2$ and let μ denote the product measure on X generated by ν. Mycielski showed that there exists a decomposition of X into two disjoint sets: one of them belongs to $\mathcal{J}_0$ and the other is of the measure μ zero and of the first category. Let

$$\mathcal{J}_\mu = \{A: \mu(A) = 0\}.$$

Since μ is a finite regular Borel measure which has no atoms, the Baire order of $\Phi(X,\mathcal{J}_\mu)$ is ω_1 (see [9], Th. 7). According to Proposition 1, the order of $\Phi(X,\mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_\mu$.

Problems. Can the condition (I) in Theorem 2 be omitted? Observe that it is possible if we add the assumption that X is locally compact. Indeed, then we put as X_0 a compact set which is a closure of an open nonempty set. The next question is: does the converse of Theorem 2 hold in this case? Saying precisely, let $\mathcal{J}$ be a σ-ideal of a locally compact perfect Polish space X and suppose that the order of $\Phi(X,\mathcal{J})$ is ω_1. We ask whether a σ-ideal $\mathcal{J}_0$ exists such that $\mathcal{J}$ is included in $\mathcal{J}_0$ and the condition (II) holds.

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