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ASYMPTOTIC BEHAVIOUR IN TIME OF SOLUTIONS TO SOME EQUATIONS GENERALIZING THE KORTEWEG-DE VRIES-BURGERS EQUATION

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Abstract: We summarize the results of a more detailed paper concerning the decay estimates for the solutions to equations describing the propagation of nonlinear waves which generalize the Korteweg-de Vries and Burgers equation.

Key words: Generalized Korteweg-de Vries and Burgers equation, propagation of nonlinear waves, decay in time of solutions.

Classification: 35Q20, 35B40

J.C. Saut has considered in [2] a class of model equations describing propagation of nonlinear waves which generalize the Korteweg-de Vries and Burgers equations. He has proved several theorems on the existence, uniqueness and regularity of solutions of the Cauchy problem for equations of the type

$$u_t + \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [f(t,u) + \delta H(x,u)] + \varepsilon Bu = g$$

where $x \in \mathbb{R}^n$, $u = u(x,t)$ is a real function, H, B are the (real) pseudodifferential operators describing dispersive and dissipative properties of the medium and f is a polynomially bounded function of u.

We prove, using the ideas of the papers [3],[4], some

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theorems on the decay in time of the solutions (in L^p norms) to one-dimensional equations of more special structure

$$(*) \quad u_t + f(u)_x + \sigma (Hu)_x + \varepsilon D^s u = 0$$

where $D^s u(x) = \int |\xi|^s \hat{u}(\xi) e^{ix\xi}\, d\xi$, $s \in \mathbb{R}^+$,

$Hu(x) = \int p(\xi)\hat{u}(\xi)e^{ix\xi}\, d\xi$ with an even positive symbol p of polynomial growth.

In the proofs we use energy inequalities for $(*)$, interpolation of Sobolev spaces and elementary properties of the fundamental solution of the linearized equation.

Theorem 1. ($\sigma \neq 0$, $\varepsilon > 0$; dispersion and dissipation effects are included)

a) If $|f'(u)| \leq C(|u|^p + 1)$ for some $p < 2(s-1)$, $s \geq 2$, $u_0 \in H^s$, then $\lim_{t \to \infty} |u(t)|_\infty = 0$.

b) The optimal decay rate (identical as for the linearized equation) is obtained assuming that f is sufficiently flat at the origin:
If also $|f'(u)| \leq C|u|^q$ for some $q > 2a + 1$ in a neighbourhood of 0 and $u_0 \in L^1$, then $|u(t)|_\infty = O((1+t)^{-1/s})$ and $|u(t)|_2 = O((1+t)^{-1/2s})$.

Theorem 2. ($\varepsilon > 0$, $\sigma = 0$; pure dissipative case)

a) The assumption in Th. 1a) plus $u_0 \in L^1$ implies that $|u(t)|_2 = O((1+t)^{-1/2s})$.

b) If $|f'(u)| \leq C|u|^q$ for some $q \geq 2s - 1$ and small $|u|$, then $|u(t)|_\infty = O((1+t)^{-1/s})$.

The pure dispersion case ($\varepsilon = 0$, $\sigma \neq 0$) leads to energetically neutral equations: $|u(t)|_2 = \text{const}$. They can have special wave-like solutions - solitons - which do not decay when t tends to infinity. Since now it is more difficult to estimate the fundamental solution of the linearized equation, we restrict our attention to the case of homogeneous symbols $p(\xi) = |\xi|^{r-1}$, $r \geq 3$, and we consider only small solutions of $(\ast)$ (with initial conditions small enough to do not support the solitons).

Theorem 3.

a) If $|f'(u)| \leq C|u|^q$, $q > r + 1$ in a neighbourhood of $u = 0$ and $|u_0|_1 + \|u_0\|_{(r-1)/2}$ is small then $|u(t)|_\infty = O((1 + |t|)^{-1/r})$ for $|t| \longrightarrow \infty$.

b) A better (than obtained by a simple interpolation) result on the decay of L^p norms of the solution is: If $q > (r + (r^2 + 4r)^{1/2})/2$ then $|u(t)|_{2(q+1)} = O((1+t)^{-(1-1/(q+1))/r})$.

The space-periodic solutions of $(\ast)$ in the case of dissipation ($\varepsilon > 0$) decay exponentially when t tends to infinity. Similarly as for the Navier-Stokes equations (cf. [1]) the solutions are asymptotically equal to solutions of the linearized equation. Namely we can prove the following

Theorem 4.

a) If $|f'(u)| \leq C(|u|^p + |u|)$ for some $p < 2(s-1)$, $s \geq 2$, then $\Lambda = \lim_{t\to\infty} (D^s u(t), u(t))/ |u(t)|_2^2$ exists and $\Lambda = \Lambda(u_0)$ is an eigenvalie of D^s.

Moreover

b) If $\sigma = 0$ then $\lim_{t\to\infty} e^{\varepsilon\Lambda t}u(t)$ exists and it is a non-zero eigenfunction of D^s corresponding to Λ.

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