COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,1 (1985)

GLOBAL (IN TIME) SOLUTION OF THE APPROXIMATE NON-LINEAR STRING EQUATION OF G. F. CARRIER AND R. NARASIMHA

Alberto AROSIO

Abstract: Known results and open problems for the local/global solvability of the nonlinear string equation proposed by G.F. Carrier and R. Narasimha.

Key words: Approximate nonlinear string equation, global/local solution.

Classification: 73D35, 34G20, 45K05

We give a brief survey on the initial-boundary value problem for a non-linear integrodifferential equation introduced by S. Bernstein [B]:

$$(1)\quad \begin{cases} u_{tt} = m\left(\int_\Omega |u_x(x,t)|^2 dx\right) \Delta_x u \text{ for } x \in \Omega\,,\ t \geq 0 \\ u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega\,,\ t \geq 0, \\ u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x). \end{cases}$$

Here Ω is an open subset of $\mathbb{R}^n$ and $m(r)$ is a continuous function such that $m(r) \geq 0$. For the choice $\Omega =]0,L[$ and

$$(2)\qquad m(r) = c_o^2 + \varepsilon r \quad (r \geq 0),$$

(1) gives the approximate model of G.F. Carrier [C] and R. Narasimha [N] for the free transversal vibration of a string

----

This paper was presented in written form on the International Spring School on Evolution Equations, Dobřichovice by Prague, May 21-25, 1984.

clamped at its ends ($c_o = \sqrt{T_o/\rho_o}$ and $\varepsilon = E/2L\,\rho_o$, where $T_o \geq 0$ is the tension in the rest position, ρ_o is the linear density of the string and E is Young's modulus).

Known results

I) Uniqueness/local existence: there exists a unique solution of (1) on some interval $[0,T[$ provided that

$$(3)\quad \begin{cases} m \text{ is a lipschitzian function with } m(r) \geq \nu > 0; \\ u_o \in H_o^1(\Omega) \cap H^{2,2}(\Omega) \text{ and } u_1 \in H_o^1(\Omega) \\ \text{(cf. [B],[D],[M],[R]).} \end{cases}$$

II) Global existence: there exists a solution of (1) on $[0,+\infty[$ in each one of the following cases:

$$(4)\quad \begin{cases} \Omega = \mathbb{R}^1;\ m \text{ is as in (2) with } c_o > 0;\ u_o \text{ and } u_1 \text{ are small} \\ (O(c_o/\sqrt{\varepsilon}\,)) \text{ in a suitable weighted } H^{2,2} \text{ space with po-} \\ \text{lynomial weight functions (cf. [GH]).} \end{cases}$$

$$(5)\quad \begin{cases} \int_0^{+\infty} m(r)\,dr \text{ is divergent; for } i = 0,1,\ \Delta_x^j u_i \in H^1(\Omega) \\ \text{for each } j \in \mathbb{N} \text{ and there exists } s>0 \text{ such that } u_i \text{ is} \\ \text{extendable to a holomorphic function on the complex} \\ \text{neighbourhood} \\ \Omega_s = \{z \in \mathbb{C}^n : d(z,\Omega) \leq s\}, \text{ with } u_i \in L^2(\Omega_s) \\ \text{(cf. [B],[P],[L],[AS],[A],[S]).} \end{cases}$$

Open questions

A) It would be interesting to exhibit a counterexample to global existence for (1) (at the present, no blow up phenomenon is explicitly known).

B) One could investigate existence/uniqueness for arbitrary initial data of finite energy, i.e. merely $u_0 \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$ (in such a generality no result is known but for the trivial case when m = const.).

References

[A] A. AROSIO: Portugaliae Math. 41(1982), to appear.

[AS] A. AROSIO, S. SPAGNOLO: Global solutions of the Cauchy problem for a nonlinear hyperbolic equation, in "Non-linear Partial Differential Equations and their Applications. Collège de France Seminar, Vol. 109", H. Brezis and J.L. Lions ed., Research Notes in Mathematics, Pitman, Boston, 1983.

[B] S. BERNSTEIN: Izv. Akad. Nauk SSSR, Ser. Mat. 4(1940), 17-26 (MR 2 # 102).

[C] G.F. CARRIER: Quart. Appl. Math. 3(1945), 157-165 and 7(1949), 97-101.

[D] R.W. DICKEY: Proc. Amer. Math. Soc. 23(1969), 459-468; Proc. Roy. Soc. Edinburgh Sect. A 82(1978/79), No. 1-2, 19-26.

[GH] J.M. GREENBERG, S.C. HU: Quart. Appl. Math. (1980), 289-311.

[L] J.L. LIONS, in "Contemporary developments in continuum mechanics and partial differential equations (G.M. De la Penha and L.A. Medeiros eds.)", North-Holland, London, 1978.

[M] G.P. MENZALA: Nonlinear Anal. 3(1979), No. 5,613-627.

[N] R. NARASIMHA: J. Sound Vibration 8(1968), 134-146.

[P] S.I. POHOŽAEV: Mat. USSR Sbornik 25,No. 1(1975), 145-158 (transl.).

[R] P.H. RIVERA RODRIGUEZ: Appl. Anal. 10(1980), 93-104.

[S] S. SPAGNOLO: Séminaire Vaillant (1982/83), Hermann, Paris.

Dipartimento di Matematica,
Via Buonarroti 2,
56100 Pisa,
Italia

(Oblatum 25.5. 1984)

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,1 (1985)

CONVERGENCE OF SOLUTIONS OF GENERALIZED KORTEWEG-DE VRIES-BURGERS EQUATIONS TO THOSE OF FIRST ORDER EQUATIONS

Piotr BILER

Abstract: We indicate the proof of the convergence of solutions of generalized Korteweg-de Vries-Burgers equations to the solutions of the limit first order equation when the parameters of the equations tend to zero.

Key words: Generalized Korteweg-de Vries-Burgers equation, propagation of nonlinear waves, convergence of solutions depending on parameters.

Classification: 35Q20, 35L60

This note deals with the convergence of solutions of one-dimensional equations describing propagation of the nonlinear waves of the type

$$(1) \quad u_t + f(u)_x + \delta(Hu)_x + \varepsilon Bu = 0$$

as δ, ε approach zero. These equations - generalizing the KdV-B equation - have been studied in [1] where, under some assumptions on the pseudodifferential operators H, B characterizing dispersive and dissipative properties of the medium and on the nonlinear flux function f, several theorems on existence, uniqueness and regularity of solutions of the Cauchy problem for (1) were proved.

This paper was presented in written form on the International Spring School on Evolution Equations, Dobřichovice by Prague, May 21-25, 1984.

We shall show that if the parameter δ is small compared to ε then there exists a subsequence of the solutions of (1) converging to a solution of the limit conservation law

(2) $\quad u_t + f(u)_x = 0.$

More precisely: we consider (1) where $|f'(u)| \le c(1 + |u|)$, $u \in \mathbb{R}$, $Bu = -u_{xx}$ (the simplest dissipative term) and $Hu(x) = -au \ (x) + \int p(\xi)\hat{u}(\xi)e^{ix\xi}\, d\xi$, $a \ge 0$ and the symbol p satisfies $0 \le p(\xi) = p(-\xi) \le C(1 + |\xi|)^{\mu}$ for some $\mu < 2$. Thus (1) is the KdV-B equation with perturbed dispersion operator ($a = 1$) and also (1) includes a class of the model wave equations with low order (<3) dispersion operator ($a = 0$).

Below $|\cdot|_p$ denotes the $L^p(\mathbb{R})$ norm, $\|\cdot\|_m$ the Sobolev space $H^m(\mathbb{R})$ norm and C denotes different inessential positive constants.

<u>Theorem</u>. Let $\Omega = \mathbb{R} \times [0,T]$, $T > 0$, and $u^{\varepsilon}_{\delta} : \Omega \longrightarrow \mathbb{R}$ be a sequence of solutions of (1) with the initial conditions $u^{\varepsilon}_{0\delta}$ satisfying $\|u^{\varepsilon}_{0\delta}\|_2 + |u^{\varepsilon}_{0\delta}|_4 \le C$.
If $\delta = o(\varepsilon^3)$, $\varepsilon \longrightarrow 0^+$, then there exists a subsequence $\{u^k\} = \{u^{\varepsilon_k}_{\delta_k}\}$ converging weakly in $L^4(\Omega)$ to u, $f(u^k) \longrightarrow f(u)$ (as distributions) and u is a solution of (2).
If in addition $f'' > 0$ then $u^k \longrightarrow u$ strongly in $L^p(\Omega)$, $1 < p < 4$.

The proof repeats the main arguments in [2], where similar facts have been proved for the classical KdV-B equation (Th. 4.1, Th. 5.1) using Tartar's compensated compactness theory.

Similarly as in [2] it suffices to show that

(3) $\{u^{\varepsilon}_{\delta}\}$ is bounded in $L^4(\Omega)$,

(4) $\quad \{\varepsilon(u^{\varepsilon}_{\delta})_x\}$,

(5) $\{\delta H u^{\varepsilon}_{\delta}\}$ are compact in $L^2(\Omega)$,

(6) $\{\varepsilon (u^{\varepsilon}_{\delta})^2_x\}$,

(7) $\{\delta (u^{\varepsilon}_{\delta})_x H u^{\varepsilon}_{\delta}\}$ are bounded in $L^1(\Omega)$.

The conditions (6) and (4) follow from the energy inequality

(8) $|u(T)|_2^2 + 2\varepsilon \int_0^T |u_x|_2^2 \le C$

obtained by taking the inner product of (1) with u and integrating in t.

Applying the multiplier $u^3 - 2\varepsilon^2 c^{-2} u_{xx}$ to (1) after some integrations by parts we arrive at the inequality

(9) $\frac{1}{4}|u|_4^4 + \varepsilon^2 c^{-2} |u_x|_2^2 + \varepsilon \int_0^T |uu_x|_2^2 + \varepsilon^3 c^{-2} \int_0^T |u_{xx}|_2^2 \le$

$$\le -3a\delta \int_0^T \int u_x u_{xx} u^2 + \delta \int_0^T \left| \int i\xi\, p(\xi) \overline{\hat{u}(\xi)} \widehat{u^3}(\xi)\, d\xi \right|.$$

The second integral on the right hand side of (9) is estimated by $C \cdot \|u\|_2^2$ using Schwarz inequality and some properties of multiplication in Sobolev spaces like Lemma 10 in [1].

If $a = 0$ then the assumption $\delta = o(\varepsilon^3)$ immediately implies (3). If $a > 0$ then a supplementary estimate is needed. Multiplying (1) by $\delta Hu + f(u)$ after rearrangements of terms and simple estimates we obtain

$$\frac{a}{2}\delta |u_x|_2^2 + \frac{\delta}{2}\int p(\xi)|\hat{u}(\xi)|^2\, d\xi + a\delta\varepsilon \int_0^T |u_{xx}|_2^2 \le$$

$$\le C + \varepsilon \int_0^T \int |f'(u)| u_x^2 + \int |F(u)| \le C(1 + |u|_\infty)$$

from (8) and assumptions on f, F, $F' = f$, and next $|u|_\infty \le$ $\le C \cdot \delta^{-1/3}$. This allows to estimate the first term on the right hand side of (9) by expressions like

$\frac{\varepsilon^3}{2} c^{-2} \int_0^T |u_{xx}|_2^2$, $\varepsilon \int_0^T |uu_x|_2^2$. Finally (3) in the case $a > 0$ is also a consequence of (9) as

(10) $\frac{1}{4}|u|_4^4 + \frac{\varepsilon^3 c^{-2}}{2} \int_0^T |u_{xx}|_2^2 \le C$.

(5) and then (7) follow from (10) and $\delta = o(\varepsilon^3)$ - observing that $|Hu|_2 \leq C \cdot \|u\|_2$.

Remark. A similar result on convergence of solutions of (1) in $L^{2(K+1)}$ with special nonlinearity $u^{2K}u_x$, $K \in \mathbb{N}$, holds if $\delta = O(\varepsilon^2)$. To see this, it suffices to multiply (1) by $\delta Hu + u^{2K+1}/(2K+1)$, integrate and recall (8).

R e f e r e n c e s

[1] J.C. SAUT: Sur quelques généralisations de l'équation de Korteweg-de Vries, J. Math. pures appl. 58(1979), 21-61.

[2] M.E. SCHONBEK: Convergence of solutions to nonlinear dispersive equations, Comm. P.D.E. 7(1982), 959-1000.

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

(Oblatum 25.5. 1984)