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LOCAL AND GLOBAL EXISTENCE AND BEHAVIOUR FOR $t \longrightarrow \infty$ OF SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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We will study nonlinear evolution equations $u'+Au+M(u) = 0$, $u(0) = \varphi$, in a Banach space B, where -A generates an analytic semigroup. Our main concern is the application of our theory to the Navier-Stokes equations.

Key words: Differential equations in abstract spaces. Special equations and problems.

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§ 1. Local existence

Let -A be the generator of an analytic semigroup e^{-tA} in a Banach space B. Let A be positive; thus the fractional powers A^{α} can be defined in the usual way. Let $\varphi \in D(A)$. Let B be reflexive. If the nonlinearity M is a mapping from D(A) into B fulfilling the Lipschitz condition

$$(I.1) \quad \|M(u)-M(v)\| \le k(C)\|A^{1-\rho}(u-v)\|,$$
$$u,v \in D(A),\ \|Au\|+\|Av\| \le C$$

for some $\rho \in (0,1)$, then there is a quantity $T(\varphi)$, $+\infty \ge T(\varphi) > 0$ with the following properties: There is a unique $u \in C^1([0,T(\varphi)),B)$ with $u(t) \in D(A)$, $0 \le t < T(\varphi)$, $Au(.) \in C^0([0,T(\varphi)),B)$,

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(I.2) $u' + Au + M(u) = 0$ on $[0,T(\varphi))$,

$$u(0) = \varphi,$$

$$\lim_{t\uparrow T(\varphi)} \|Au(t)\| = +\infty \text{ if } T(\varphi) < +\infty.$$

This result has been stated by Kato [K] . A proof has been given by the author (not yet published). It rests on the consideration of the weak derivative of $M(u)$. u has the additional property that $u'(t) \in D(A^{1-\rho})$, $0 < t < T(\varphi)$. For our purposes it is important to study (I.2) with initial values from B only (instead of $D(A)$). The possibility of solving (I.2) with initial values from B depends on the growth of M. We want to present two theorems:

<u>Theorem I.1:</u> <u>Let</u> $\varphi \in B$. <u>For some</u> $\rho_1 \in (0,1)$ <u>let</u> M <u>be a mapping from</u> $D(A^{1-\rho_1})$ <u>into</u> B <u>fulfilling the Lipschitz condition</u>

(I.3) $$\|M(u)-M(v)\| \le c[(\|A^{1-\rho_1}u\|+\|A^{1-\rho_1}v\|)\|u-v\|+(\|u\|+\|v\|)\|A^{1-\rho_1}(u-v)\|]$$

<u>Then there exists a quantity</u> $T(\varphi)$, $+\infty \ge T(\varphi) > 0$ <u>with the following properties</u>: <u>There is unique</u> $u \in C^0([0,T(\varphi)),B)$ <u>with</u> $u(t)\ D(A^{1-\rho_1})$, $0 < t < T(\varphi)$, $A^{1-\rho_1}u \in C^0((0,T(\varphi)),B)$, $t^{1-\rho_1}A^{1-\rho_1}u(t)$ <u>bounded on</u> $(0,T)$, $T<T(\varphi)$, $u(0)=\varphi$,

(I.4) $$u(t) = e^{-tA}\varphi - \int_0^t e^{-(t-s)A}M(u(s))\,ds,$$

$$\lim_{t\uparrow T(\varphi)} \|u(t)\| = +\infty \text{ if } T(\varphi) < +\infty.$$

<u>From</u> (I.4) <u>it follows that</u> $u(t) \in D(A)$, $0 < t < T(\varphi)$, $Au \in C^0((0,T(\varphi)),B)$, $u \in C^1((0,T(\varphi)),B)$,

$$u' + Au + M(u) = 0 \text{ on } (0,T(\varphi)).$$

For a <u>proof</u> see [W1, IV].

This theorem already covers some applications to the Navier-Stokes Equations but we still need a version which turns out to be somewhat stronger, at least as it concerns the Navier-Stokes Equations.

It is based on the consideration of the equation $(A^{-\delta}u)' + AA^{-\delta}u + A^{-\delta}M(A^{\delta}A^{-\delta}u) = 0$ or $w' + Aw + A^{-\delta}M(A^{\delta}w) = 0$ for some $\delta > 0$. Solving the latter equation one can consider the resulting solution w as a weak solution of $u' + Au + M(u) = 0$ and try to improve on the regularity of u. This device was used by Fujita and Kato [FK] to treat the Navier-Stokes Equations.

Theorem I.2: Let M be a mapping from $D(A^{1-\rho_1})$ into B for some $\rho_1 \in (0,1)$. We set

$$\tilde{M}(u) = M(A^{\delta}u),$$

$u \in D(A^{1-\rho_1+\delta})$ and assume that the following Lipschitz inequalities hold:

$$\|A^{-\delta}\tilde{M}(u)\| \le c\|A^{1-\rho_1'}u\|^2(\|A^{\delta}u\|+1),$$

$$\|A^{-\delta}(\tilde{M}(u)-\tilde{M}(v))\| \le c[(\|A^{\delta}u\|+\|A^{\delta}v\|+1)(\|A^{1-\rho_1'}u\|+\|A^{1-\rho_1'}v\|)\cdot \|A^{1-\rho_1'}(u-v)\| + (\|A^{1-\rho_1'}u\|^2+\|A^{1-\rho_1'}v\|^2+1)\cdot \|A^{\delta}(u-v)\|]$$

for some $\delta > 0$ and some $\rho_1' \in (0,1)$ with

$$0 < 1-2\rho_1' \le \delta < 1-\rho_1'.$$

Then for any $\varphi \in B$ there exists a quantity $T(\varphi)$, $+\infty \ge T(\varphi) > 0$ such that there is a unique

$$u \in C^0([0,T(\varphi)),B)$$

with

$$u(t) \in \bigcap_{0<\varepsilon'<1} D(A^{1-\varepsilon'-\delta}), t^{1-\varepsilon-\delta}A^{1-\varepsilon-\delta}u(t) \text{ bounded on } (0,T), T<T(\varphi),$$

$$A^{1-\varepsilon'-\delta}u \in C^0((0,T(\varphi)),B),\ 0 < \varepsilon' < 1,$$

$$A^{-\delta}u(t) = e^{-tA}A^{-\delta}\varphi - \int_0^t e^{-(t-s)A}A^{-\delta}M(A^{\delta}A^{-\delta}u(s))\,ds.$$

<u>If</u> $T(\varphi) < +\infty$ <u>then</u>

$$\lim_{t\uparrow T(\varphi)} \|u(t)\| = \lim_{t\uparrow T(\varphi)} \|A^{\delta}A^{-\delta}u(t)\| = +\infty$$

<u>or</u>

$$\omega(\tilde{\delta},T) = \sup_{0\le t\le\tilde{\delta},\,T+t<T(\varphi)} \|t^{1-\rho_1'-\delta}A^{1-\rho_1'-\delta}e^{-tA}u(T)\|$$

<u>does not converge uniformly in</u> $T\in[0,T(\varphi))$ <u>to</u> 0 <u>for</u> $\tilde{\delta}\to 0$.

<u>Proof:</u> We sketch the proof. The details can be found in [W1, Sätze IV.9, IV.11]. First we observe that $t^{\alpha}A^{\alpha}e^{-tA}x \to 0$ for $t\to 0$ and for any $\alpha > 0$. We consider the mapping

$$Tw(t) = e^{-tA}A^{-\delta}\varphi - \int_0^t e^{-(t-s)A}A^{-\delta}M(A^{\delta}w(s))\,ds$$

on the complete metric space

$$\begin{aligned}\mathfrak{M}_{\tilde{T}} = \Big\{ w \,|\, & w\in C^{0}([0,\tilde{T}],D(A^{\delta})),\ w(0) = A^{-\delta}\varphi,\\ & w\in C^{0}((0,\tilde{T}],D(A^{1-\varepsilon'})),\\ & \cdot^{1-\delta-\varepsilon'}A^{1-\varepsilon'}w(\cdot)\in L^{\infty}((0,\tilde{T}),B),\\ & \|A^{\delta}w(t)\| \le \|A^{\delta}e^{-\cdot A}A^{-\delta}\varphi\|_{L^{\infty}((0,\tilde{T}))} + 1,\\ & \|t^{1-\delta-\varepsilon'}A^{1-\varepsilon'}w(t)\| \le 2\|\cdot^{1-\delta-\varepsilon'}A^{1-\varepsilon'}e^{-\cdot A}A^{-\delta}\varphi\|_{L^{\infty}((0,\tilde{T}))},\\ & 0 < t \le \tilde{T}\Big\},\ 0 < \varepsilon' < 1-\delta,\end{aligned}$$

endowed with the metric

$$\begin{aligned}\mu_{\tilde{T}}(v_1,v_2) = {} & \sup_{0\le t\le\tilde{T}} \|A^{\delta}(v_1(t)-v_2(t))\| + \\ & + \sup_{0<t<\tilde{T}} \|t^{\frac{1}{2}-\varepsilon'}A^{1-\varepsilon'}(v_1(t)-v_2(t))\|.\end{aligned}$$

We want to show that by T the space $\mathfrak{M}_{\tilde{T}}$ is mapped into itself if $\tilde{T}$ is sufficiently small and $\varepsilon' = \rho_1$. We have

$$\| t^{1-\delta-\varepsilon'} A^{1-\varepsilon'} Tw(t) \|$$

$$\leq \| t^{1-\delta-\varepsilon'} A^{1-\delta-\varepsilon'} e^{-tA} \varphi \| +$$

$$+ ct^{1-\delta-\varepsilon'} \int_0^t \frac{e^{-c(t-s)}}{|t-s|^{1-\varepsilon'}} \| A^{1-\rho_1'} w(s) \|^2 (\| A^\delta w(s) \| + 1)\, ds,$$

$$\leq \| t^{1-\delta-\varepsilon'} A^{1-\delta-\varepsilon'} e^{-tA} \varphi \| +$$

$$+ ct^{1-\delta-\varepsilon'} \int_0^t \frac{1}{|t-s|^{1-\varepsilon'}} \cdot \frac{1}{s^{2(1-\rho_1'-\delta)}}\, ds \cdot$$

$$\cdot 4 \sup_{0<s\leq t} \| s^{1-\delta-\varepsilon'} A^{1-\varepsilon'} e^{-sA} A^{-\delta} \varphi \|^2 \cdot (\| A^\delta w(s) \| + 1)\, ds.$$

Since $\int_0^t \frac{1}{|t-s|^{1-\varepsilon'}} \cdot \frac{1}{s^{2(1-\rho_1'-\delta)}}\, ds = \int_0^1 \frac{1}{|t-s|^{1-\rho_1'}} \cdot \frac{1}{s^{2(1-\rho_1'-\delta)}}\, ds =$

$= \frac{c}{t^{2-3\rho_1-2\delta}} \leq \frac{c}{t^{1-\delta-\rho_1}}$ for small t if $\varepsilon' = \rho_1'$, $1-2\rho_1' \leq \delta$ we have proved that $\| t^{1-\delta-\rho_1'} A^{1-\rho_1'} Tw(t) \|$ can be bounded in the same way as $\| t^{1-\delta-\rho_1'} A^{1-\rho_1'} w(t) \|$. The contraction property of T is proved in the same way. The size of $\tilde{T}$ depends on $\| A^\delta A^{-\delta} \varphi \|$ and the smallness of $\omega(\tilde{\delta}, 0)$. Thus the last part of Theorem I.2 follows. The regularity properties of u are standard.

There are results corresponding to the just described ones for equations $u'(t) + A(t)u(t) + M(u(t)) = f(t)$ if the domain of definition $D(A(t))$ of the closed operators $A(t)$ is time independent (for details see [W1, IV.]). One can also improve on the regularity of u in Theorems I.1, I.2 if M is an analytic mapping between suitable Banach spaces as it is the case for the Navier-Stokes Equations (for details see [W1, II., IV. and VI.]).

§ 2. Local strong solutions of the Navier-Stokes Equations

Velocity u and pressure π of a viscous incompressible fluid under the influence of an external force f are supposed to be determinated by the Navier-Stokes Equations

$$\text{(II.1)} \quad \frac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla u + \nabla\pi = f, \qquad \nabla\cdot u = 0.$$

This equation is considered over a cylindrical domain $(0,T)\times\Omega \subset \mathbb{R}^{n+}$ where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary. In the physically relevant case $n=3$, Ω is the space domain being filled out by the fluid. ν is the viscosity. The n-vector $u\cdot\nabla u$ is defined to have the components $\sum_{i=1}^{n} u_i \frac{\partial u_\lambda}{\partial x_i}$, $1 \le \lambda \le n$; we prescribe boundary values: $u(t,x)\,|\,\partial\Omega = 0$, $t > 0$, $u(0,x) = \varphi(x)$. This problem is subsumed under the theory of nonlinear evolution equations in the following way: First $(L^p(\Omega))^n$, $p > 1$ (in what follows we omit the exponent n) is decomposed into the direct sum

$$L^p(\Omega) = H_p(\Omega) + \{\nabla g \,|\, \nabla g \in L^p(\Omega)\},$$

where $H_p(\Omega)$ is the completion of the divergence free $C_0^\infty(\Omega)$-vector fields with respect to the $L^p(\Omega)$-norm (see [FM]); $H_p(\Omega)$ is then reflexive and the "projection" of $L^p(\Omega)$ on $H_p(\Omega)$ is a bounded operator, which we denote by P_p or simply P. Applying (formally) P_p to (II.1) and assuming that because of $\nabla\cdot u = 0$ the equality $u = P_p u$ holds, we get

$$\text{(II.2)} \quad u' + Au + P_p(u\cdot\nabla u) = P_p f, \qquad u(0) = 0$$

with $A = A_p = -\nu P_p \Delta$. We also set $M(u) = P_p(u \cdot \nabla u)$.

Because of its mathematical interest we will consider (II.1), (II.2) in any number of space dimensions. The domain of definition of A is $H^{2,p}(\Omega) \cap \overset{\circ}{H}{}^{1,p}(\Omega) \cap H_p(\Omega)$. As it has been proved in [W1, Gi1] A is a positive operator in the Banach space $B = H_p(\Omega)$ which generates an analytic semigroup e^{-tA} with exponential decay. As for the fractional powers A^α it was proved by Giga [Gi2] that

(II.3) $D(A^\alpha) = D((-\Delta)^\alpha) \cap H_p(\Omega)$, $0 \le \alpha \le 1$,

with equivalent graph norms; $-\Delta$ is the usual Laplacian with domain of definition $H^{2,p}(\Omega) \cap \overset{\circ}{H}{}^{1,p}(\Omega)$. In particular this means that

$$D(A^\alpha) = \overset{\circ}{H}{}^{2\alpha,p}(\Omega) \cap H_p(\Omega), \quad 0 \le \alpha \le \tfrac{1}{2}, \quad \alpha \neq \tfrac{1}{2p},$$

$$D(A^{\frac{1}{2p}}) \subset \overset{\circ}{H}{}^{\frac{1}{p},p}(\Omega) \text{ with a continuous imbedding.}$$

Here $H^{s,p}(\Omega)$ are the complex interpolation spaces between the Sobolev spaces $H^{k,p}(\Omega) = W^{k,p}(\Omega)$ of integer order k. $\overset{\circ}{H}{}^{s,p}(\Omega)$ is the completion of $C_o^\infty(\Omega)$ in the norm of $H^{s,p}(\Omega)$. (II.2) is then considered as a nonlinear evolution equation in $B = H_p(\Omega)$ to which we want to apply our results of § 1. Once having solved (II.2) the pressure is determined by

$$\nabla\pi = (I-P_p)\nu\Delta u - (I-P_p)u \cdot \nabla u + (I-P_p)f.$$

Our results depend on p and we start with

a) $p > \frac{n}{3}$. Then the Lipschitz condition (I.1) is fulfilled as can be easily proved by an application of Sobolev's imbedding Theorems. Thus (II.2) can be solved locally in t if $\varphi \in D(A)$. It will turn out that for $n = 3$ the exponent $p = \frac{5}{4}$ is important.

b) $p > n$, say $p = n + \varepsilon$. We have the trivial estimate

$$\|M(u)\| \le c\|u\cdot\nabla u\|_{L^p(\Omega)},$$

$$\le c\|\nabla u\|_{C^0(\bar{\Omega})}\|u\|_{H_p(\Omega)}.$$

From (II.3) and our assumption $p > n$ it can be derived that

$$D(A^{1-\rho}) \subset C^{1+\alpha}(\bar{\Omega})$$

for some $\rho, \alpha \in (0,1)$ (see [W1, VI.]). Thus we have

$$\|M(u)\| \le c\|A^{1-\rho}u\|\,\|u\|.$$

Since there is also a corresponding Lipschitz estimate, we can apply Theorem I.1 (see [W1, VI.], [W2]).

Thus we get a local strong solution for any $\varphi \in B = H_p(\Omega)$ if $p > n$; since this solution has a bounded $C^{1+\alpha}(\bar{\Omega})$-norm it is not surprising that it is classical for $0 < t < T(\varphi)$ provided f is sufficiently regular.

c) $p = n$. We want to apply Theorem I.2. We choose $\delta = \frac{1}{2}$. Partial integration shows that

$$\|u\cdot\nabla u\|_{H^{-1,n}(\Omega)} \le c\|u\|^2_{L^{2n}(\Omega)}.$$

Using Giga's result on the fractional powers of $A = A_n$ and Sobolev theorem we arrive at

$$\|A_n^{-\frac{1}{2}}M(u)\|_{H_n(\Omega)} \le c\|A_n^{\frac{1}{4}}u\|^2_{H_n(\Omega)}.$$

It is easily shown that also a corresponding Lipschitz condition holds. Thus we see that with $\rho_1' = \frac{1}{4}$ we have

$$0 < 1-2\rho_1' = \frac{1}{2} = \delta;$$

therefore Theorem I.2 is applicable and gives a solution of the integral equation

$$A_n^{-\frac{1}{2}}u(t) = e^{-tA_n}A_n^{-\frac{1}{2}}\varphi - \int_0^t e^{-(t-s)A_n}A_n^{-\frac{1}{2}}P_n(u\cdot\nabla u)(s)\,ds + \\ + \int_0^t e^{-(t-s)A_n}f(s)\,ds$$

on $[0,T(\varphi))$ for every $\varphi \in B = H_n(\Omega)$. This solution can be identified with the solution constructed in b) on $(0,T(\varphi))$. Thus u is regular, its degree of regularity depending on f.

It may be noted that the solutions constructed in a),b),c) are in $D(A_p^{1+\frac{1}{2p}-\varepsilon})$ on $(0,T(\varphi))$, $0 < \varepsilon$, since $u\cdot\nabla u$ depends analytically on the components of u and ∇u (for details see [W1, VI]).

§ 3. Global questions: The connection between weak solutions and local strong solutions

It turns out that in § 2, c) the quantity $T(\varphi)$ is finite if u is not uniformly continuous on $[0,T(\varphi))$ as a mapping from $[0,T(\varphi))$ into $H_n(\Omega)$. Thus in what follows the $H_n(\Omega)$-norm of u(t) plays a major role.

As it is well known there is also an access to the Navier-Stokes equations via the notion of a weak solution.

Definition III.1.: Let $f \in L^2((0,T),H^{-1,2}(\Omega))$, $\varphi \in H_2(\Omega)$. An element $u \in L^\infty((0,T),L^2(\Omega)) \cap L^2((0,T),\overset{\circ}{H}{}^{1,2}(\Omega))$, which is weakly continuous from $[0,T]$ into $L^2(\Omega)$ and which fulfills $\nabla\cdot u(t) = 0$, a.e., is called a weak solution of (II.1) over $(0,T)\times\Omega$ if

$$\text{(III.1)} \quad -\int_0^T (u,\psi')\,dt + \nu\int_0^T (\nabla u,\nabla\psi)\,dt + \int_0^T (u\cdot\nabla u,\psi)\,dt \\ = (u(0),\psi(0)) + \int_0^T (f,\psi)\,dt$$

for all testing functions $\psi \in C^1([0,T],\overset{\circ}{H}{}^{1,n}(\Omega))$ *with* $\nabla\cdot\psi(t) = 0$ *on* $[0,T]$, $\psi(T) = 0$.

For a discussion of this definition and the determination of the pressure π see [L, 1.6]. It goes back to E. Hopf that such a weak solution exists for all $T > 0$, i.e. on $(0,\infty)\times\Omega$; it can be constructed via Galerkin's approximation procedure, and a weak solution constructed in this way has an important additional property, namely:

$$\text{(III.2)} \quad \|u(t)\|^2 + 2\nu \int_r^t \|\nabla u(\sigma)\|^2 \, d\sigma \le \|u(r)\|^2 + 2 \int_r^t (f(\sigma),u(\sigma)) \, d\sigma,$$

for almost all $r \ge 0$ and all $t \ge r$ ($\|.\|$ is the $L^2(\Omega)$-norm, $(.,.)$ the $L^2(\Omega)$-scalar product). (III.2) is called "energy inequality". It does not follow from (III.1) since it is not allowed to insert u as a testing function. For details see [L, 1.6].

The weak solutions with energy inequality play a distinct rôle because under some additional assumptions a uniqueness theorem for them holds, namely:

Theorem III.1: Let φ, f *be as in Definition* III.1. *Let* u_1, u_2 *be weak solutions of* (II.1) *in the sense of Definition* III.1. *Let* (III.2) *be valid for* $r = 0$ *and all* t, $0 \le t \le T$. *Let one of the* u^i, $i = 1,2$, *say* u^1, *fulfil the condition*

$$\text{(III.3)} \quad u^1 \in L^{r'}((0,T),L^r(\Omega))$$

with $n < r < +\infty$, $2 < r' < +\infty$, $\frac{2}{r'}+\frac{n}{r} = 1$, *or*

$$\text{(III.4)} \quad u^1 \in C^0([0,T],L^n(\Omega)).$$

Then $u^1(t) = u^2(t)$, $0 \le t \le T$.

The proof of the uniqueness under the condition (III.3) is due to Serrin [S], under the condition (III.4) to Sohr and von Wahl [SW].

In particular (III.4) means that $\varphi \in H_n(\Omega)$.

As for (III.3) the condition $n < r$ could be weakened somewhat:

Theorem III.2: Let $\varphi \in H_n(\Omega)$, $f \in C^{\alpha}([0,T],L^{n+\delta}(\Omega))$ for some $\alpha \in (0,1)$ and some $\delta > 0$. Let u^1,u^2 be weak solutions in the sense of Definition III.1. Let u^1 fulfill (III.2) for all r,t, $0 \le r \le t \le T$. Let $u^2 \in L^{\infty}((0,T),L^n(\Omega))$. Then $u^1(t) = u^2(t)$ for all $t \in [0,T]$.

A proof can be found in [SW]. As it was proved in [SW] too any weak solution $u \in L^{\infty}((0,T),L^n(\Omega))$ with φ,f as in Theorem III.2. fulfills (III.2) for all r,t, $0 \le r \le t \le T$. Moreover, if u_1,u_2 are weak solutions being in $L^{\infty}((0,T),L^n(\Omega))$ with data φ,f as in Theorem III.2, then $u_1(t) = u_2(t)$. This is an easy consequence of Theorem III.2 and was proved in [SW]. Thus $L^{\infty}((0,T),L^n(\Omega))$ is a uniqueness class for weak solutions which was previously not known (cf. e.g. [L, 1.6]). It is clear now that any weak solution u with $u(t) \in L^n(\Omega)$ a.e. and with (III.2) for almost all $r \in (0,T)$ and all t, $r \le t \le T$, may be reconstructed locally in t with the aid of Theorem I.2 and § 2, c). The result is a generalization of Leray's famous structure theorem.

Theorem III.3: We assume that $f \in L^2((0,\infty),L^n(\Omega))$ and that $f \in C^{\alpha}([0,T], L^{n+\delta}(\Omega))$ for all $T > 0$ with α,δ as in Theorem III.2. Let u be a weak solution of (II.1) for all T with (III.2) for almost all $r \ge 0$ and all $t \ge r$. Let $u \in L^2((0,+\infty),L^n(\Omega))$. Then it follows:

1) On $[T_0,+\infty)$ u is regular, where T_0 is sufficiently large.

2) $[0,T_0) = \bigcup_{\nu=1}^{\infty} J_{\nu} \cup S$, where J_{ν} are pairwise disjoint open intervalls on which u is regular and where S has measure 0. S is called the singular set of u since $S \cap J_{\nu} = \emptyset$, $\nu = 1,2,\dots$.

Let us make some remarks on the proof: III.3, 1) follows from the fact that in § 2, c) the quantity $T(\varphi)$ is $+\infty$ if $\|\varphi\|_{H_n(\Omega)}$ is sufficiently small and if $f \in L^2((0,\infty),L^n(\Omega))$. This is caused by the exponential decay of the semigroup $e^{-A_n t}$. As for III.3, 2) this assertion follows from reconstructing $u(t)$ on an interval $[r, r+\varepsilon(r)]$ with $\varepsilon(r) > 0$ according to Theorem III.1, (III.4) and § 2, c). Here r is a point such that $u(r) \in L^n(\Omega)$ and (III.2) holds for all $t \geq r$.

Since Galerkin's approximation procedure gives us a weak solution with the desired properties if $n = 3,4$ we have proved the existence of weak solutions with III.3.,1), III.3.,2) in these cases.
For $n = 3,4$ we have

$$\text{(III.5)}\quad u(t) \in \overset{\circ}{H}{}^{1,2}(\Omega) \subset L^6(\Omega) \quad \text{a.e.}, \; n = 3,$$

$$u(t) \in \overset{\circ}{H}{}^{1,2}(\Omega) \subset L^4(\Omega) \quad \text{a.e.}, \; n = 4.$$

(III.5) shows that in the case $n = 3$ also § 2, b) may be sufficient to prove the structure theorem. In fact, for $n = 3$ we do not need at all the construction of local strong solutions with "bad initial values". It can be proved that

$$\text{(III.6)}\quad u \cdot \nabla u \in L^{5/4}((0,T) \times \Omega)$$

for any weak solution. According to Solonnikov's potential theoretical estimates for the linear equations $u' - \nu\Delta u + \nabla\pi = f$, $\nabla \cdot u = 0$ [So] it follows that $u \in L^{5/4}((0,T), H^{2,5/4}(\Omega))$, provided $f \in L^{5/4}((0,T), L^{5/4}(\Omega))$, $\varphi \in H_2(\Omega)$ + some modest degree of differentiability. Thus $u(t) \in H^{2,5/4}(\Omega)$ a.e. and since $5/4 > \frac{n}{3} = 1$ we can simply apply II.a) to reconstruct the weak solution locally in t if u fulfils the energy inequality (III.2). (III.6) was proved in [La], its generalization to arbitrary n and its consequences were considered in [W3]. The conditions on f in Theorems III.2, III.3 are partially caused by our interpretation of "regular", but they are not weakest possible: We mean by "regular" that $u(t) \subset H^{2,n+\delta}(\Omega) \subset C^{1+\beta}(\bar{\Omega})$ for some $\beta \in (0,1)$. The weak solutions constructed in Theorem III.3 frequently are called <u>turbulent solutions</u>.

Remark: A weak solution u with $u \in L^{r'}((0,T),L^r(\Omega))$, $\frac{2}{r'}+\frac{n}{r} = 1$, $n \leq r < +\infty$, $2 < r' \leq +\infty$, satisfies (III.2) for all r,t, $0 \leq r \leq t \leq T$ ([S], [SW]). This will be freely used in the sequel.

We now study the regularity of a weak solution u in the sense of Definition III.1. As it is natural, this question is connected with the uniqueness of u. Serrin [S] has proved that any weak solution $u \in L^{r'}((0,T),L^r(\Omega))$ with

(III.7) $\frac{2}{r'}+\frac{n}{r} < 1$,

$n < r \leq +\infty$, $2 < r' \leq +\infty$,

is C^2 in x provided f is sufficiently regular. In fact u is a classical solution ([W2]). Sohr [So] has weakened Serrin's condition to

(III.8) $\frac{2}{r'}+\frac{n}{r} = 1$,

$n < r < +\infty$, $2 < r' < +\infty$, $n=3,4$.

In the last time some attention has been given to the case $r = n$, $r' = +\infty$. From the remark at the beginning of this paragraph and Theorem III.1 it follows that $u \in C^0([0,T],L^n(\Omega))$ is also sufficient for regularity: The weak solution then can be reconstructed as a strong one according to § 2, c) which is regular and for which $T(\varphi) = T$ since the uniform continuity of the strong solution on $[0,T(\varphi))$ follows from the coincidence of this solution with the weak solution in question. A different proof was given in [W4]. Sohr [So] has proved that certain subclasses of $L^\infty((0,T),L^n(\Omega))$ also imply regularity, and in [W3] it was proved that the stability of a weak solution in $L^\infty((0,T),L^n(\Omega))$ implies its regularity.

If we simply assume that $u \in L^\infty((0,T),L^n(\Omega))$ the question whether if u is regular or not is still open, but in [SW] the following weaker theorem was proved:

Theorem III.4: Let u be a weak solution in the sense of Definition III.1. Let

$u \in L^\infty((0,T),L^n(\Omega)) \cap L^p((0,+\infty),L^n(\Omega))$ for some $p \geq 2$ and all $T, T>0$.

Let φ, f be as in Theorem III.3. Then (according to our Remark above) the structure theorem holds, and the singular set S can be characterised as follows:

1. S is at most countable,
2. $t \in S$ if and only if u is continuous in t from the right with respect to the $L^n(\Omega)$-norm but discontinuous from the left.

u is continuous in t from the right with respect to the $L^n(\Omega)$-norm for any $t \geq 0$.

The proof rests of course on the reconstruction of u as a strong solution with the aid of § 2, c), the remark at the beginning of § 3 and Theorem IV.1.

In a recent preprint Giga [Gi3] has stated results similar to Theorem III.3 but as far as it could be seen his methods are different from [SW].

We have not dealt with the case $n = 2$ since it is well known then that any weak solution is regular and therefore unique.

§ 4. Global questions: The behaviour of weak solutions for $t \to \infty$.

We will be brief at this point and concentrate on the case $n = 3$. The energy inequality (III.2) suggests very strongly that there may be some sort of decay for u if the assumptions on f are appropriate. It is well known (see [M]) that under a suitable integrability condition on $\|f(t)\|_{L^2(\Omega)}$, $\|f'(t)\|_{L^2(\Omega)}$ over $(0,\infty)$ any weak solution over $(0,+\infty)\times\Omega$ in the sense of Definition III.1 with (III.2) fulfills the estimate

$$\|\nabla u(t)\|_{L^2(\Omega)} \le \frac{c}{t^{1/4}}$$

for large t.

Now let g be a C^1-function with $|g'(t)/g(t)| \to 0, t\to\infty, g>0$, let u be a weak solution as above, then

$$\text{(IV.1)} \quad \|\nabla u(t)\|_{L^2(\Omega)} \le \frac{c}{g(t)}, \text{ t large.}$$

There is also an additional condition concerning the integrability of $\|g(t)f(t)\|_{L^2(\Omega)}$ over $(0,\infty)$ which we have omitted here. (IV.1) was proved by Sohr [So]. He first shows that $\int_t^\infty \|u(\sigma)\|^{r'}_{L^r(\Omega)} d\sigma < \infty$, t large, for some r,r' with $3 = n < r < +\infty$, $2 < r' < +\infty$, $\frac{2}{r'}+\frac{3}{r} = \frac{2}{r'}+\frac{n}{r} = 1$. Then he uses a variant of Solonnikov's estimates [Sol], namely:

$$\text{(IV.2)} \quad \int_t^T \|(gu')\|^2_{L^2(\Omega)} d\tilde{t} + \int_t^T \|\Delta gu\|^2_{L^2(\Omega)} d\tilde{t}$$

$$\le c\left\{\|\nabla g(t)u(t)\|^2_{L^2(\Omega)} + \int_t^T \|gf\|^2_{L^2(\Omega)} d\tilde{t} + \right.$$

$$\left. + \int_t^T \|g'u\|^2_{L^2(\Omega)} d\tilde{t}\right\} e^{c\int_t^T \|u(\tilde{t})\|^{r'}_{L^r(\Omega)} d\tilde{t}}, \text{ t large, } t \le T < +\infty.$$

In fact for the exponent 2, (IV.2) is easily derived, but under suitable assumptions (IV.2) also holds for exponents $q \ge 2$ and in higher dimensions (with some modifications for the norm of the initial value); if a term $\int_t^T \|gu\|^2_{L^2(\Omega)} d\tilde{t}$ or $\int_t^T \|gu\|^q_{L^q(\Omega)} d\tilde{t}$ is added on the right side then (IV.2) remains valid for exterior domains. For details see [So].

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