COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,1 (1985)

SOME REGULARITY RESULTS FOR QUASI-LINEAR PARABOLIC SYSTEMS

Michael STRUWE

Abstract: Regularity results for quasilinear parabolic equations and systems recently obtained by Giuquinta, Vivaldi, and this author are surveyed. The presentation allows immediate extension to variational inequalities.

Key-words: Quasilinear parabolic system, regularity

Classification: 35K55

Let Ω be a domain in R^n, $T>0$, $Q_T = \Omega \times [0,T]$. Denote by L^p, $H^{m,p}$, etc. the usual Lebesgue and Sobolev spaces. In particular,

$$V = L^2([0,T], H^{1,2}(\Omega; R^N)) \cap L^\infty(Q_T; R^N)$$

denotes bounded and measurable functions $u : Q_T \to R^N$ such that

$$\int_0^T \int_\Omega |\nabla u(x,t)|^2 dxdt < \infty \quad ,$$

where $\nabla u^i = \left(\frac{\partial}{\partial x_1} u^i, \ldots, \frac{\partial}{\partial x_n} u^i\right)$ is the spatial derivative of u^i and $u = (u^1, \ldots, u^N)$.

In this survey we shall be concerned with the regula-

This paper was presented on the International Spring School on Evolution Equations, Dobřichovice by Prague, May 21-25, 1984 (invited lecture).

rity of weak solutions $u \in V$ of quasilinear parabolic systems

$$(0.1) \qquad \partial_t u^i - \partial_\alpha\left(a^{ik}_{\alpha\beta}\,\partial_\beta u^k\right) = f^i(\cdot,u,\nabla u)\;, \qquad 1 \leq i \leq N$$

in the sense that for all $\varphi \in C^\infty_0(Q_T;R^N)$ [1)]

$$(0.2) \qquad -\int_{Q_T} u^i\varphi^i_t\,dxdt + \int_{Q_T} a^{ik}_{\alpha\beta}\partial_\beta u^k\partial_\alpha\varphi^i\,dxdt = \int_{Q_T} f^i(\cdot,u,\nabla u)\,\varphi^i\,dxdt\,.$$

We assume that (0.1) is uniformly parabolic in the sense that the $a^{ik}_{\alpha\beta} \in L^\infty(Q_T)$ satisfy the condition

$$(0.3) \qquad a^{ik}_{\alpha\beta}(x,t)\,\xi^i_\alpha\xi^k_\beta \geq \lambda|\xi|^2$$

for all $\xi \in R^{n\times N}$ and almost every $(x,t) \in Q_T$, with a uniform constant $\lambda > 0$. Moreover we suppose that $f : Q_T \times R^N \times R^{n\times N} \to R^N$ is a Carathéodory function and satisfies the growth estimate

$$(0.4) \qquad |f(x,t,u,p)| \leq a|p|^2 + b$$

a.e. in $Q_T \times R^N \times R^{n\times N}$, with constants $a,b \in R$.

Given a solution u of (0.2) we denote $M = |u|_\infty$. Finally remark that by density (0.2) also holds for φ in

$$\mathcal{T} = H^{1,2}_0([0,T];\,L^2(\Omega;R^N)) \cap L^2([0,T];\,H^{1,2}_0(\Omega;R^N)) \cap L^\infty(Q_T;R^N)\,.$$

In the sequel, under suitable conditions relating λ, a, and M, we derive partial regularity in the interior of Q_T of weak solutions to (0.1). For diagonal systems $\left(\text{i.e. } a^{ik}_{\alpha\beta} = a_{\alpha\beta}\,\delta^{ik}\,,\;\; \delta^{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}\right)$ we obtain

---

1) Repeated Roman indeces by convention are summed from 1 to N, Greek indeces from 1 to n. Moreover, for brevity $\varphi_t = \frac{\partial}{\partial t}\varphi$, $\partial_\alpha\varphi = \frac{\partial}{\partial x_\alpha}\varphi$.

Hölder continuity in the interior. From here onwards higher regularity is obtained in standard manner, cp. [16].

Our presentation summarizes results from [12], [13], [20], [21], [22]. [1)] These results are well-known in the elliptic (or stationary) case. The difficulty only consists in conveying the methods.
Therefore in the following we first review elliptic regularity theory for a particularly simple example of a system of type (0.1). In the second main chapter the parabolic analogues of fundamental estimates for solutions of elliptic systems will be derived. Of course, we may concentrate on those estimates where in the time-dependent case significant changes have to be made with respect to the stationary case.

Even though our results well confirm the general expectation that any result for elliptic system of type (0.1) will (with appropriate modifications) carry over to the parabolic case there are regularity problems for evolution equations that possess no stationary equivalent. Thus for these problems elliptic regularity theory does not provide any intuition. Some open problems of this kind will be mentioned at the end of this paper.

1. To motivate what we consider "basic estimates" we consider a weak solution $u \in H^{1,2} \cap L^{\infty}(\Omega;\mathbb{R}^N)$, $\|u\|_{\infty} = M$, of an elliptic system

$$(1.1) \qquad -\Delta u^i = f^i(\cdot,u,\nabla u) \ , \quad 1 \leq i \leq N \ ,$$

with quadratic growth with respect to the gradient

$$(1.2) \qquad |f(x,u,p)| \leq a|p|^2 \ .$$

Of the numerous results and methods for treating such prob-

1) For reference we point out two more recent articles [2], [23] on regularity theory for parabolic systems and inequalities.

lems we shall indicate a method by Giaquinta-Giusti-Modica (cp. [7], [8]) and the hole-filling technique of Hildebrandt-Widman [14].

By definition u weakly solves (1.1) iff

$$(1.3)\quad \int_\Omega [\nabla u^i \nabla\varphi^i - f^i(x,u,\nabla u)\varphi^i]dx = 0 \qquad \forall \varphi \in H_0^{1,2} \cap L^\infty(\Omega;\mathbb{R}^N).$$

Step 1 in the elliptic case consists in observing that for any $\eta \in C_0^\infty(\Omega)$ the function $u\eta$ is admissible as a testing function in (1.3).

Step 2: Let $x_0 \in \Omega$, $r > 0$ satisfy $B_{2r} = B_{2r}(x_0) \subset \Omega$. Choose $\tau \in C_0^\infty(B_{2r})$ satisfying $0 \leq \tau \leq 1$, $\tau \equiv 1$ on B_r $|\nabla\tau| \leq \frac{c}{r}$, and let $\bar{u} = \fint_{B_{2r}\setminus B_r} u\,dx$. [1] Inserting

$\varphi = (u-\bar{u})\tau^2$ in (1.3) yields:

Caccioppoli's inequality: Suppose $2aM < \lambda$, then [2]

$$(1.4)\quad \int_{B_r} |\nabla u|^2 dx \leq \frac{c}{r^2} \int_{B_{2r}\setminus B_r} |u-\bar{u}|^2 dx .$$

Step 3 simply consists in applying the

Poincaré-Sobolev-inequality: Let $2^+ = \frac{2n}{n+2} < 2$, then

$$(1.5)\quad \fint_{B_{2r}\setminus B_r} |u-\bar{u}|^2 dx \leq cr^2 \left(\fint_{B_{2r}\setminus B_r} |\nabla u|^{2^+} dx \right)^{2/2^+} \leq cr^2 \fint_{B_{2r}\setminus B_r} |\nabla u|^2 dx.$$

Step 4 combines (1.4), (1.5) to obtain an

Inverse Hölder inequality: Let $2aM < \lambda$, then

$$(1.6)\quad \fint_{B_r} |\nabla u|^2 dx \leq c \left(\fint_{B_{2r}} |\nabla u|^{2^+} dx \right)^{2/2^+} .$$

---

1) $\fint$ denotes mean value.

2) The letter c denotes a generic constant depending only on the data $a, M, \lambda, \ldots$

In Step 5 a variant [8, Proposition 1.1, p. 122], [11, Proposition 5.1] of "Gehring's lemma" [6] is employed to derive from (1.6) the

L^p-estimate: Let $2aM < \lambda$, then $u \in H^{1,p}_{loc}(\Omega; R^N)$ for some $p > 2$ and

$$(1.7) \quad \fint_{B_r} |\nabla u|^p dx \leq c\left(\fint_{B_{2r}} |\nabla u|^2 dx\right)^{p/2} .$$

Step 6: Let $v \in u + H^{1,2}_o \cap L^\infty(B_r; R^N)$ satisfy $\Delta v = 0$. Recall the Campanato estimate [3] for v

$$(1.8) \quad \int_{B_\varrho} |\nabla v|^2 dx \leq c\left(\frac{\varrho}{r}\right)^n \int_{B_r} |\nabla v|^2 dx , \quad \forall \varrho < r$$

and the maximum principle: $\|v\|_\infty \leq \|u\|_\infty$.

Subtract $\Delta v = 0$ from (1.1) and test with $u-v$ to obtain

$$(1.9) \quad \fint_{B_r} |\nabla (u-v)|^2 dx \leq a \fint_{B_r} |\nabla u|^2 |u-v| dx$$

$$\leq c\left(\fint_{B_r} |u-v|^2 dx\right)^{1-2/p}\left(\fint_{B_r} |\nabla u|^p dx\right)^{2/p}$$

$$\leq c\left(r^{2-n} \int_{B_r} |\nabla u|^2 dx\right)^{1-\frac{2}{p}} \fint_{B_{2r}} |\nabla u|^2 dx$$

Together, (1.8), (1.9) yield

$$(1.10) \quad \int_{B_\varrho} |\nabla u|^2 dx \leq c\left[\left(\frac{\varrho}{r}\right)^n + \left(r^{2-n}\int_{B_r} |\nabla u|^2 dx\right)^{1-\frac{2}{p}}\right]\int_{B_r} |\nabla u|^2 dx .$$

If $\liminf_{r \to o} r^{2-n} \int_{B_r(x_o)} |\nabla u|^2 dx < \varepsilon_o$ is sufficiently small the usual iteration procedure [8, Lemma 2.1, p. 86] therefore gives

$$(1.11)\qquad \int_{B_\varrho(x_o)} |\nabla u|^2 dx \leq c r^{n-2+2\alpha}$$

for some $\alpha > 0$. Hence from [18, 3.5.2] we obtain

Partial regularity: Suppose $2aM < \lambda$, then there exists an open set $\Omega_o \subset \Omega$ such that u is Hölder continuous on Ω_o (even $C^{1,\alpha}$), and

$$(1.12)\qquad \Omega \setminus \Omega_o \subset \{x_o \,|\, \liminf_{r\to o} r^{2-n} \int_{B_r(x_o)} |\nabla u|^2 dx \geq \varepsilon_o\} \quad .$$

Hence also the $(n-2-\varepsilon)$-dimensional Hausdorff measure

$$\mathcal{H}^{n-2-\varepsilon}(\Omega \setminus \Omega_o) = 0 \,,$$

for all $\varepsilon < p-2$, cp. [8, Theorem 2.1, p. 100].

For diagonal systems like (1.1) complete regularity may be obtained under the condition $aM < \lambda$ cp. [10], [15], [25]. If $2aM < \lambda$, this fact may be directly inferred from the above partial regularity result and the following estimatedue to Giaquinta and Giusti (cp. [10]).

Step 7: For any $\varepsilon, \sigma, R > 0$ there exists k such that for some $r \in]\sigma^k R, R]$

$$(1.13)\qquad r^{2-n} \int_{B_r} |\nabla u|^2 dx \leq \varepsilon \;.$$

A different approach to regularity for diagonal systems consists in the hole-filling technique [14].

Step 8: For $x_o \in \Omega$, $\varrho > 0$ let $G^\varrho = G^\varrho(\cdot, x_o)$ be a mollified Green's function satisfying

$$(1.14)\qquad \int_\Omega \nabla w \nabla G^\varrho dx = \fint_{B_\varrho(x_o)} w \, dx \,, \qquad \forall w \in H_o^{1,2}(\Omega) \quad .$$

As $\varrho \to 0$ $G^\varrho \rightharpoonup G$ weakly in $H^{1,q}(\Omega)$, for any $q < \frac{n}{n-1}$, where G is the Green's function for $-\Delta$ on Ω.

Harnack inequality: With constants $c_i = c_i(\mathrm{dist}(x_o, \partial\Omega))$ there holds ($B_r = B_r(x_o)$, etc.):

$$r^{2-n} \leq c_1 \sup_{B_{2r}\setminus B_{r/2}} G \leq c_2 \inf_{B_{2r}\setminus B_{r/2}} G \leq c_3 r^{2-n} . \tag{1.15}$$

Moreover, since $-\Delta G = 0$ in $B_{2r}\setminus\{x_o\}$, for any $w \in H_o^{1,2}(B_{2r})$:

$$\int_{B_{2r}\setminus B_r} |\nabla G|^2 |w|^2 dx \leq c \int_{B_{2r}\setminus B_{r/2}} G^2 |\nabla w|^2 dx + \frac{c}{r^2} \int_{B_{2r}\setminus B_{r/2}} G^2 w^2 dx; \tag{1.16}$$

this estimate moreover also holds for G^ϱ, provided $\varrho < r/2$,

Step 9 consists in deriving the following weighted Caccioppoli inequality: Suppose $2aM < \lambda$, then for any $\delta > 0$, $\varrho < r$, τ as in Step 2:

$$\begin{aligned} &⨍_{B_\varrho} |u-\bar{u}|\tau^2 dx + \int_{B_{2r}} |\nabla u|^2 G^\varrho \tau^2 dx \\ &\leq \delta \int_{B_{2r}\setminus B_r} |\nabla G^\varrho|^2 |u-\bar{u}|^2 \tau^2 dx + \frac{c}{\delta r^2} \int_{B_{2r}\setminus B_r} |u-\bar{u}|^2 dx + \\ &+ \frac{c}{r^2} \int_{B_{2r}\setminus B_r} |u-\bar{u}|^2 G^\varrho dx \end{aligned} \tag{1.17}$$

which is obtained on inserting $\varphi = (u-\bar{u})G^\varrho\tau^2$ into (1.3).

Conclusion: Estimating the first term on the right of (1.17) by (1.16), we may let $\varrho \to 0$. Choosing $\delta = \sup^{-1}_{B_{2r}\setminus B_{r/2}} \{G(x,x_o)\}$ and applying estimates (1.15), (1.5), from (1.17) we then obtain the following estimate for the function $\Phi(r) = \int_{B_r} |\nabla u|^2 G\, dx$:

$$\Phi(r/2) \leq c_1[\Phi(2r) - \Phi(r/2)] . \tag{1.18}$$

Adding c_1 times the left-hand-side to (1.18) and dividing by (c_1+1) we infer that for some $\mu < 1$

$$\Phi(r/2) \leq \mu\, \Phi(2r)$$

i.e. by iteration:

$$r^{2-n} \int_{B_r(x_o)} |\nabla u|^2 dx \leq c\,\Phi(r) \leq cr^{2\alpha}$$

for some $\alpha > 0$, uniformly in $x_o \in \Omega$, $2r \leq \mathrm{dist}(x_o, \partial\Omega)$ with c depending only on the data and $\mathrm{dist}(x_o, \partial\Omega)$; i.e. $u \in C^\alpha$. Higher regularity may now be obtained by standard techniques [17], [18]. •

2. The results of the preceding section clearly will carry over to parabolic systems of type (0.1) if Steps 1-9 may be performed (with appropriate modifications) in this case.

For $r > 0$ introduce $\Lambda_r =]-r^2, 0[$, $B_r = B_r(0)$, $Q_r = B_r \times \Lambda_r$: the standard parabolic cylinder centered at $(0,0)$ with boundary $\partial Q_r = (B_r \times \{0\}) \cup S_r$. Since we shall only be concerned with interior regularity for (0.1) and since we are free to shift the origin in $\mathbb{R}^{n+1}$, it will suffice to state estimates on such standard regions $Q_r \subset Q$ indiscriminately denoting the transposed domain Q_T by Q.

In the following u will always denote a weak solution of a system like (0.1), (0.3), (0.4). For simplicity we consider only the case $b = 0$. Step 1 is by no means obvious in the parabolic case, since $V \neq \mathcal{T}$. An argument as in [16, Lemma III. 4.1] however shows that testing functions like $\varphi = u \cdot \eta \cdot 1_{]-\infty,0[}$ [1] may be inserted into (0.2).

Lemma 2.1: $u \in C^0(]0,T[; L^2_{loc}(\Omega; \mathbb{R}^N))$, and for any function $\eta \in C^\infty(\mathbb{R}^{n+1})$ vanishing in a neighborhood of S_r there holds:

$$\frac{1}{2} \int_{B_r \times \{0\}} |u|^2 \eta dx + \int_{Q_r} [a^{ik}_{\alpha\beta} \partial_\alpha u^i \partial_\beta u^k \eta - f^i(\cdot, u, \nabla u) u^i \eta] dxdt$$

---

1) 1_A denotes the characteristic function of a set A.

(2.1) $$= \frac{1}{2}\int_{Q_r}[|u|^2\eta_t - 2a^{ik}_{\alpha\beta}\partial_\beta u^k u^i \partial_\alpha \eta]dxdt .$$

Proof: Let $\psi_\varrho(t)$ be a mollifier, $u_\varrho = u*\psi_\varrho$, etc. Define $1^\varrho(t) \equiv 1$ if $t \leq -\varrho$, $1^\varrho(t) = -\frac{t}{\varrho}$, if $-\varrho \leq t \leq 0$, $1^\varrho(t) \equiv 0$ if $t > 0$. Then $\varphi = \varphi^{(\varrho)} = (u_\varrho \eta 1^\varrho)_\varrho \in \mathcal{T}$ and

$$\int_Q u[(u_\varrho\eta 1^\varrho)_\varrho]_t dxdt = -\frac{1}{2}\int_{Q_r}[|u_\varrho|^2]_t \eta 1^\varrho dxdt$$

$$= \frac{1}{2}\int_{Q_r}|u_\varrho|^2\eta_t 1^\varrho dxdt - \frac{1}{2}\int_{Q_r}|u_\varrho|^2\eta|1^\varrho_t|dxdt .$$

Hence from (0.2)

$$\frac{1}{2}\fint_{-\varrho}^{0}\int_{B_r\times\{t\}}|u_\varrho|^2\eta dxdt + \int_{Q_r}\Big[(a^{ik}_{\alpha\beta}\partial_\beta u^k)_\varrho\partial_\alpha u^i_\varrho \eta 1^\varrho -$$
$$- (f^i(\cdot,u,\nabla u))_\varrho u^i_\varrho \eta 1^\varrho\Big]dxdt$$

$$= \frac{1}{2}\int_{Q_r}\Big[|u_\varrho|^2\eta_t 1^\varrho - 2(a^{ik}_{\alpha\beta}\partial_\beta u^k)_\varrho u^i_\varrho \partial_\alpha \eta 1^\varrho\Big]dxdt ,$$

and (2.1) follows on letting $\varrho \to 0$. qed.

Step 2: Choose $\tau \in C^\infty(\mathbb{R}^{n+1})$ vanishing on S_{2r} and such that $0 \leq \tau \leq 1$, $\tau \equiv 1$ on Q_r, $|\nabla\tau|^2 + |\tau_t| \leq \frac{c}{r^2}$, $\sup_{B_{2r}\times\{t\}} \tau \leq c \fint_{B_{2r}\times\{t\}} \tau\, dx$, and let

$$\bar{u}(t) = \int_{B_{2r}\times\{t\}} u\tau^2 dx \Big/ \int_{B_{2r}\times\{t\}} \tau^2 dx \quad (:= 0, \text{ if } \tau \equiv 0 \text{ on } B_{2r}\times\{t\}).$$

Choosing $\varphi = (u-\bar{u}(t))\tau^2 1_{]-\infty,0[}$ and taking account of (2.1) yields the

Caccioppoli type inequality: Suppose $2aM < \lambda$. Then

(2.2) $$\int_{Q_r}|\nabla u|^2 dxdt \leq \frac{c}{r^2}\int_{\Lambda_{2r}}\left(\int_{B_{2r}\times\{t\}}|u-\bar{u}(t)|^2 dx\right)dt .$$

Proof: Note that $\bar{u}(t)$ is absolutely continuous. (Insert $\tau^2 1_{]-\infty,t[}$ into (0.2).) Therefore by Fubini's theorem

$$\int_{Q_{2r}} \partial_t \bar{u}(u-\bar{u})\,\tau^2 dxdt = \int_{\Lambda_{2r}} \partial_t \bar{u}\left(\int_{B_{2r}\times\{t\}} (u-\bar{u})\,\tau^2 dx\right) dt = 0 ,$$

and (2.1) - with $(u-\bar{u})$ instead of u - implies (2.2).

qed.

Step 3. Although there is no general equivalent of (1.5) for functions in V, for *solutions* of systems like (0.1) several Poincaré-Sobolev type inequalities can be stated. A preliminary observation is needed. Let $\chi \in C_o^\infty(B_r)$ satisfy $0 \leq \chi \leq 1$, $\sup \chi \leq c \fint_{B_r} \chi\, dx$ and let $B_\chi = \operatorname{supp} \chi$, $\tilde{u}_\chi = \fint_{B_\chi} u\, dx$, $\bar{u}_\chi = \int_{B_\chi} u\chi\, dx / \int_{B_\chi} \chi\, dx$. Then

$$(2.3)\qquad \int_{B_\chi} |u-\tilde{u}_\chi|^2 dx \leq \int_{B_\chi} |u-\bar{u}_\chi|^2 dx \leq c \int_{B_\chi} |u-\tilde{u}_\chi|^2 dx$$

Proof of (2.3). The first inequality expresses the minimizing property of the mean:

$$\int_{B_\chi} |u-\tilde{u}_\chi|^2 dx \leq \int_{B_\chi} |u-c|^2 dx , \quad \forall c \in \mathbb{R} .$$

To obtain the second we estimate

$$\int_{B_\chi} |u-\bar{u}_\chi|^2 dx = \int_{B_\chi} \left| \int_{B_\chi} [(u(x)-\tilde{u}_\chi) + (\tilde{u}_\chi - u(y))]\chi(y)\,dy \right|^2 dx \left(\int_{B_\chi} \chi dx\right)^{-2}$$

$$\leq \left(1 + \frac{\sup \chi}{\fint_{B_\chi} \chi dx}\right)^2 \int_{B_\chi} |u-\tilde{u}_\chi|^2 dx . \qquad \text{qed.}$$

Poincaré-Sobolev-type estimates: Let $\tau, \bar{u}$ be as in Step 2, $2^+ = \frac{2n}{n+2}$. For any solution u of (0.2)-(0.4) there holds

$$(2.4)\qquad \sup_{t\in\Lambda_r} \int_{B_r\times\{t\}} |u-\bar{u}(t)|^2 dx \leq c \int_{Q_{2r}} |\nabla u|^2 dxdt .$$

Moreover, for any $\varepsilon > 0$

(2.5) $$\fint_{Q_r} |u-\bar{u}(t)|^2 dxdt \leq \varepsilon r^2 \fint_{Q_{2r}} |\nabla u|^2 dxdt + c(\varepsilon) r^2 \left(\fint_{Q_{2r}} |\nabla u|^{2^+} dxdt \right)^{2/2^+} .$$

Finally, if $\tilde{u} = \fint_{Q_r} u \, dxdt$, $\tilde{\tilde{u}} = \fint_{Q_{2r} \setminus Q_{r/2}} u \, dxdt$

(2.6.a) $$\int_{Q_r} |u-\tilde{u}|^2 dxdt \leq cr^2 \int_{Q_{2r}} |\nabla u|^2 dxdt$$

(2.6.b) $$\int_{Q_{2r} \setminus Q_{r/2}} |u-\tilde{\tilde{u}}|^2 dxdt \leq cr^2 \int_{Q_{4r} \setminus Q_{r/4}} |\nabla u|^2 dxdt .$$

<u>**Proof:**</u> Introduce $\tilde{u}(t) = \fint_{B_r \times \{t\}} u \, dx$.

Let $t_o \in \Lambda_r$ satisfy

$$\int_{B_r \times \{t_o\}} |u-\bar{u}(t_o)|^2 dx = \sup_{t \in \Lambda_r} \int_{B_r \times \{t\}} |u-\bar{u}(t)|^2 dx .$$

Testing (0.1) with $\varphi = (u-\bar{u}(t)) \tau^2 1_{]-\infty, t_o[}$, the "elliptic" Poincaré inequality (1.5) , (2.1) , (2.3) and a reasoning as in the proof of (2.1) yield

$$\int_{B_r \times \{t_o\}} |u-\bar{u}(t_o)|^2 dx \leq c \int_{Q_{2r}} |\nabla u|^2 dxdt + \frac{c}{r^2} \int_{Q_{2r}} |u-\bar{u}(t)|^2 dxdt$$

$$\leq c \int_{Q_{2r}} |\nabla u|^2 dxdt + \frac{c}{r^2} |u-\tilde{u}(t)|^2 dxdt \leq c \int_{Q_{2r}} |\nabla u|^2 dxdt .$$

This proves (2.4) .

By (2.4) , (2.3) and the "elliptic" Sobolev-Poincaré inequality

$$\fint_{Q_r} |u-\bar{u}(t)|^2 dxdt \leq \sup_{t \in \Lambda_r} \left(\fint_{B_r \times \{t\}} |u-\bar{u}(t)|^2 dx \right)^{1-\frac{2^+}{2}}$$

$$\cdot \fint_{\Lambda_r} \left(\fint_{B_r \times \{t\}} |u-\bar{u}(t)|^2 dx \right)^{\frac{2^+}{2}} dt$$

$$\leq c\left(r^2 \fint_{Q_{2r}} |\nabla u|^2 dxdt \right)^{1-\frac{2^+}{2}} \left(r^{2^+} \fint_{Q_{2r}} |\nabla u|^{2^+} dxdt \right),$$

and (2.5) is a consequence of Young's inequality.

To prove (2.6.a) let $\vartheta \in C_o^\infty(B_{2r})$ satisfy $0 \leq \vartheta \leq 1$, $\vartheta \equiv 1$ on B_r, $|\nabla \vartheta| \leq \frac{c}{r}$, and for any $s \in \Lambda_{2r} \setminus \Lambda_r$ let $t_s \geq s$ satisfy

$$\int_{B_{2r}\times\{t_s\}} |u-\tilde{u}(s)|^2 \vartheta^2 dx = \sup_{t \geq s} \int_{B_{2r}\times\{t\}} |u-\tilde{u}(s)|^2 \vartheta^2 dx .$$

Testing (0.1) with $\varphi = (u-\tilde{u}(s))\vartheta^2 1_{]s,t_s[}$ (s fixed!) like (2.1) from Young's inequality we obtain

$$\fint_s^{t_s} \left(\int_{B_{2r}\times\{t\}} |u-\tilde{u}(s)|^2 \vartheta^2 dx \right) dt \leq \int_{B_{2r}\times\{t_s\}} |u-\tilde{u}(s)|^2 \vartheta^2 dx \leq$$

$$\leq \int_{B_{2r}\times\{s\}} |u-\tilde{u}(s)|^2 \vartheta^2 dx + c(\delta) \int_{Q_{2r}} |\nabla u|^2 dxdt +$$

$$+ \frac{\delta}{r^2} \int_s^{t_s} |u-\tilde{u}(s)|^2 \vartheta^2 dxdt .$$

Choosing $\delta < \frac{1}{8}$ and noting that $t_s - s < 4r^2$ the last term on the right is dominated by the left hand side of this inequality. Applying the Poincaré inequality (1.5) to the first term on the right and averaging with respect to $s \in \Lambda_{2r} \setminus \Lambda_r$ there hence results

$$\fint_{\Lambda_{2r} \setminus \Lambda_r} \int_{B_{2r}\times\{t_s\}} |u-\tilde{u}(s)|^2 \vartheta^2 dxds \leq c \int_{Q_{2r}} |\nabla u|^2 dxdt$$

and (2.6.a) is a consequence of (2.3) and the estimate

$$\int_{Q_r} |u-\tilde{u}|^2 dxdt \leq \int_{Q_r} |u-\tilde{u}(s)|^2 dxdt \leq r^2 \int_{B_{2r}\times\{t_s\}} |u-\tilde{u}(s)|^2 \vartheta^2 dx ,$$

for $s \in \Lambda_{2r} \setminus \Lambda_r$.

To obtain (2.6.b) introduce $\chi(x) = \vartheta(8x)$ and perform the above calculations with $\vartheta(1-\chi)$ instead of ϑ, $\tilde{\tilde{u}}(t) = \fint_{B_r\times\{t\}\setminus B_{r/4}\times\{t\}} u\, dx$ instead of $\tilde{u}(t)$. qed.

Step 4 combines estimates (2.2), (2.5) to obtain

Inverse Hölder inequality: Suppose $2a\,M < \lambda$, then, for any $\varepsilon > 0$

(2.7) $$\fint_{Q_r} |\nabla u|^2 dxdt \leq c(\varepsilon)\left(\fint_{Q_{4r}} |\nabla u|^{2^+} dxdt\right)^{2/2^+} + \varepsilon \fint_{Q_{4r}} |\nabla u|^2 dxdt \,.$$

Step 5. A Gehring-type lemma [13, Proposition 1.3] now yields the

L^p-estimate: Suppose $2a\,M < \lambda$, then $|\nabla u| \in L^p_{loc}(Q_T)$ for some $p > 2$ and

(2.8) $$\left(\fint_{Q_r} |\nabla u|^p dxdt\right)^{1/p} \leq c\left(\fint_{Q_{4r}} |\nabla u|^2 dxdt\right)^{1/2} .$$

Step 6. Suppose that $a^{ik}_{\alpha\beta}(x,t) = A^{ik}_{\alpha\beta}(x,t,u(x,t))$ and that the A^{ik} are uniformly Hölder continuous. Let $\mathring{A} = A(0,\tilde{u})$ (omitting indeces) and let $v \in V$ solve

(2.9) $$\partial_t v^i - \partial_\alpha \mathring{A}^{ik}_{\alpha\beta} \partial_\beta v^k = 0 \text{ in } Q_r \,, \; v = u \text{ on } S_r \,.$$

Recall the Campanato estimate for v (see [4])

(2.10) $$\int_{Q_\varrho} |\nabla v|^2 dxdt \leq c\left(\frac{\varrho}{r}\right)^{n+2} \int_{Q_r} |\nabla v|^2 dxdt \,, \quad \forall \varrho < r \,.$$

It is unknown whether there holds a general maximum principle for (2.9). In the case considered here, however, it is possible to show that

(2.11) $$\sup_{Q_r} |v| \leq c \cdot M \,,$$

cp. [20, (3.7)]. Substracting (2.9)

from (0.1) and testing with $(u-v)\,1_{]-\infty,0[}$ like (2.1) we obtain

(2.12) $$\lambda \int_{Q_r} |\nabla(u-v)|^2 dxdt \leq a \int_{Q_r} |\nabla u|^2 |u-v|\, dxdt + \int_{Q_r} |A-\mathring{A}|\, |\nabla u|\, |\nabla(u-v)|\, dxdt \,,$$

whence in particular $\int_{Q_r} |\nabla(u-v)|^2 dxdt \leq c \int_{Q_r} |\nabla u|^2 dxdt$.

Estimating $|A-\mathring{A}| \leq \omega(r, |u-\tilde{u}|^2)$ with a concave function ω, and applying (2.8) and Jensen's inequality we obtain from (2.12), (2.11)

$$\fint_{Q_r} |\nabla(u-v)|^2 dxdt \leq c\Big(\fint_{Q_r} |\nabla u|^p dxdt\Big)^{2/p} \cdot$$

(2.13) $$\cdot\Big[\Big(\fint_{Q_r} \omega\, dxdt\Big)^{1-(2/p)} + \Big(\fint_{Q_r} |u-v|^2 dxdt\Big)^{1-(2/p)}\Big]$$

$$\leq c\Big[\omega\Big(r, \fint_{Q_r} |u-\tilde{u}|^2 dxdt\Big)^{1-(2/p)} + \Big(r^2 \fint_{Q_r} |\nabla u|^2 dxdt\Big)^{1-(2/p)}\Big] \cdot$$

$$\cdot \fint_{Q_{4r}} |\nabla u|^2 dxdt .$$

Together, (2.10) and (2.13) yield for all $\varrho < r$

(2.14) $$\int_{Q_\varrho} |\nabla u|^2 dxdt \leq c\Big[\Big(\frac{\varrho}{r}\Big)^{n+2} + \chi(r)\Big] \int_{Q_{4r}} |\nabla u|^2 dxdt$$

with

$$\chi(r) = \omega\Big(r, \fint_{Q} |u-\tilde{u}|^2 dxdt\Big)^{1-(2/p)} + \Big(r^2 \fint_{Q_r} |\nabla u|^2 dxdt\Big)^{1-(2/p)} .$$

By (2.6) therefore, if $\liminf_{r\to o} r^2 \fint_{Q_r(x_o,t_o)} |\nabla u|^2 dxdt < \varepsilon_o$

is sufficiently small, the usual iteration procedure ([8, Lemma 2.1, p. 86]) yields that

$$\int_{Q_r(x_o,t_o)} |\nabla u|^2 dxdt \leq cr^{n+2\alpha}$$

for some $\alpha > 0$. Hence from (2.6) and [5, Theorem 3.1] we have

Partial regularity: Suppose $2aM < \lambda$, then there exists an open set $\tilde{Q}_o \subset Q$ such that u (and ∇u) is Hölder continuous on $\tilde{Q}$ and

(2.15) $$Q \setminus \tilde{Q} \subset \{(x_o,t_o) \mid \liminf_{r\to o} r^2 \fint_{Q_r(x_o,t_o)} |\nabla u|^2 dxdt \geq \varepsilon_o\} .$$

Hence also the $(n-\varepsilon)$-dimensional Hausdorff measure with respect to the metric $\delta((x_1,t_1),(x_2,t_2)) = \max\{|x_1-x_2|, |t_1-t_2|^{1/2}\}$

$$\mathfrak{H}^{n-\varepsilon}(Q\setminus\tilde{Q},\delta) = 0$$

for all $\varepsilon < p-2$, cp. [13. Proposition 3.2].

Now we specialize to diagonal systems with coefficients $a_{\alpha\beta}^{ik} = a_{\alpha\beta}\delta^{ik}$, $a_{\alpha\beta} \in L^{\infty}(Q)$.[1)] Due to anisotropy of space-time step 7 cannot be conveyed to parabolic systems immediately. A different weight function has to be employed like the fundamental solution to the heat equation in R^{n+1}.

Therefore we turn to the parabolic analogue of the hole-filling-technique, presented in [20] and further applied in [12], [22].

Step 8: For $(x_o,t_o) \in Q_T$, $\varrho > 0$ let $G^{\varrho} = G^{\varrho}(\cdot,(x_o,t_o))$ be a mollified Green's function of the operator $\mathcal{L} = \partial_t - \partial_\alpha(a_{\alpha\beta}(\cdot)\partial_\beta\cdot)$ satisfying

$$(2.16) \qquad \int_{Q_T} [w_t G^{\varrho} + a_{\alpha\beta}\partial_\beta w \partial_\alpha G^{\varrho}]dxdt = \int_{Q_\varrho(x_o,t_o)} w \quad dxdt$$

for all $w \in \mathcal{T}$ vanishing on the time like boundary $S = \partial\Omega \times [0,T] \cup \Omega \times \{0\}$ of Q_T. As $\varrho \to 0$ $G^{\varrho} \longrightarrow G$ as a distribution and uniformly outside a neighborhood of (x_o,t_o), where G is the Green's function for $\mathcal{L}$ on Q_T, cp. [1]. The Harnack inequality for parabolic equations [19], [24] implies that on $Q_r^{\varepsilon} = \{(x,t) \in Q_r | t < -\varepsilon r^2\}$

$$(2.17) \qquad \sup_{Q_r^{\varepsilon}(x_o,t_o)} G \leq c_1(\varepsilon) \inf_{Q_r^{\varepsilon}(x_o,t_o)} G$$

with constants depending on ε and $\operatorname{dist}((x_o,t_o),S)$. Moreover the estimate [1; Theorems 7,8]

$$(2.18) \qquad \begin{aligned} c_2|t-t_o|^{n/2}\exp\left(-c_3\frac{|x-x_o|^2}{|t-t_o|}\right) &\leq G((x,t),(x_o,t_o)) \\ &\leq c_4|t-t_o|^{n/2}\exp\left(-c_5\frac{|x-x_o|^2}{|t-t_o|}\right) \end{aligned}$$

for $t < t_o$, with constants again depending on $\operatorname{dist}((x_o,t_o),S)$ shows that there exists a function $\nu(\varepsilon) \to 0$ $(\varepsilon \to 0)$ such that

$$(2.19) \qquad \sup_{Q_{2r} \setminus (Q_{r/2} \cup Q_{2r}^{\varepsilon})} G \leq \nu(\varepsilon) \inf_{Q_{2r}} G(\cdot,(0,r^2)) ,$$

---

1) $\delta^{ik} = 0$ if $i \neq k$, $\delta^{ik} = 1$ if $i = k$.

where for convenience we have again shifted (x_o,t_o) to $(0,0)$.

Again due to presence of time-derivatives an analogue of (1.16) only will hold for solutions of (0.1) in general. We return to this point below.

In the following we will state estimates again on our standard domains Q_r . To facilitate notation let $G_\theta = G(\cdot,(0,\theta))$.

<u>Step 9</u>. Suppose $2aM < \lambda$. Let τ be as in Step 2, $\tilde{\tilde{u}} = \fint_{Q_{2r}\setminus Q_{r/2}} u\,dxdt$. Then for any $\delta > 0$, $\theta > 0$ we obtain the following

<u>weighted Caccioppoli inequality</u>:

$$\int_{Q_{2r}} |\nabla u|^2 G_\theta \tau^2 dxdt \leq \frac{c}{r^2} \int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 G_\theta dxdt +$$

(2.20)

$$+ \delta \int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 |\nabla G_\theta|^2 G_\theta^{-3/2} \tau^2 dxdt + \frac{c}{\delta r^2} \int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 G_\theta^{3/2} dxdt.$$

<u>Proof</u>: Let $\varrho \in]0,\sqrt{\theta}[$. Testing (0.1) with $\varphi = (u-\tilde{\tilde{u}}) G_\theta^\varrho \tau^2 1_{]-\infty,0[}$, by (2.1) and using Green's identity (2.16) we obtain

$$\int_{Q_{2r}} \left[a_{\alpha\beta}\partial_\alpha u^i \partial_\beta u^i - f^i(\cdot,u,\nabla u)(u-\tilde{\tilde{u}})^i\right] G_\theta^\varrho \tau^2 dxdt$$

$$\leq \frac{1}{2}\int_{Q_{2r}} \left[|u-\tilde{\tilde{u}}|^2 (G_\theta^\varrho \tau^2)_t - a_{\alpha\beta}\partial_\beta |u-\tilde{\tilde{u}}|^2 \partial_\alpha (G_\theta^\varrho \tau^2)\right] dxdt$$

$$= \frac{1}{2}\int_{Q_{2r}} \left[|u-\tilde{\tilde{u}}|^2 \tau^2 (G_\theta^\varrho)_t - a_{\alpha\beta}\partial_\beta\left(|u-\tilde{\tilde{u}}|^2\tau^2\right)\partial_\alpha G_\theta^\varrho\right] dxdt$$

$$+ \int_{Q_{2r}} \Big[|u-\tilde{\tilde{u}}|^2 G_\theta^\varrho \tau_t \tau - a_{\alpha\beta}\partial_\beta |u-\tilde{\tilde{u}}|^2 \partial_\alpha \tau G_\theta^\varrho \tau +$$

$$+ a_{\alpha\beta}\partial_\beta \tau |u-\tilde{\tilde{u}}|^2 \partial_\alpha G_\theta^\varrho \tau\Big] dxdt$$

$$\leq \frac{c}{\varepsilon r^2}\int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 G_\theta^\varrho dxdt + \varepsilon \int_{Q_{2r}} |\nabla u|^2 G_\theta^\varrho \tau^2 dxdt$$

$$+ \frac{c}{\delta r^2} \int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 (G_\theta^\varrho)^{3/2} dxdt + \delta \int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 |\nabla G_\theta^\varrho|^2 (G_\theta^\varrho)^{-3/2} \tau^2 dxdt$$

Since $2aM < \lambda$ the second term on the right is dominated by the left hand side, if ε is sufficiently small. By Fatou's lemma we may then let $\varrho \to 0$ to obtain (2.20). qed.

Estimating the second term on the right of (2.20) requires an additional .

Step 10. For any $\theta > 0$ (without any smallness assumption relating a, M, and λ)

$$\int_{Q_{2r}\setminus Q_r} |u-\tilde{\tilde{u}}|^2 |\nabla G_\theta|^2 G_\theta^{-3/2} \tau^2 dxdt \leq$$

$$\leq c \int_{Q_{2r}\setminus Q_{r/2}} |\nabla u|^2 G_\theta^{1/2} dxdt + \frac{c}{r^2} \int_{Q_{2r}\setminus Q_{r/2}} |u-\tilde{\tilde{u}}|^2 G_\theta^{1/2} dxdt. \tag{2.21}$$

Proof: Let $\sigma(x,t) = \tau(2x,2t)$, $\eta = \tau(1-\sigma)$. Testing the equations (2.16), (0.2) with $w = G_\theta^{-1/2} |u-\tilde{\tilde{u}}|^2 \eta^2 1_{]-\infty,0[}$, resp. $\varphi = 4(u-\tilde{\tilde{u}}) G_\theta^{1/2} \eta^2 1_{]-\infty,0[}$ and subtracting, by (2.1) we obtain:

$$-2 \int_{B_{2r}\times\{0\}} |u-\tilde{\tilde{u}}|^2 G_\theta^{1/2} \eta^2 dx + 4 \int_{Q_{2r}} |u-\tilde{\tilde{u}}|^2 G_\theta^{1/2} \eta_t \eta \, dxdt$$

$$- \frac{1}{2} \int_{Q_{2r}} a_{\alpha\beta} \partial_\alpha G_\theta \partial_\beta G_\theta G_\theta^{-3/2} |u-\tilde{\tilde{u}}|^2 \eta^2 dxdt$$

$$-4 \int_{Q_{2r}} \left[a_{\alpha\beta} \partial_\alpha u^i \partial_\beta u^i - f^i(\cdot, u, \nabla u)(u-\tilde{\tilde{u}})^i \right] G_\theta^{1/2} \eta^2 dxdt$$

$$+4 \int_{Q_{2r}} \left[a_{\alpha\beta} \partial_\alpha G_\theta^{1/2} \partial_\beta \eta |u-\tilde{\tilde{u}}|^2 \eta - a_{\alpha\beta} \partial_\beta |u-\tilde{\tilde{u}}|^2 \partial_\alpha \eta G_\theta^{1/2} \eta \right] dxdt = 0 .$$

Estimating the last term by Young's inequality we obtain (2.21). qed.

Conclusion. (2.19) - (2.21) together now yield the following estimate for the function $\Phi_\theta(r) = \int_{Q_r} |\nabla u|^2 G_\theta dxdt$:

$$\Phi_\theta(\tfrac{r}{4}) \leq \delta \int_{Q_{2r}\setminus Q_{r/2}} |\nabla u|^2 G_\theta^{1/2} dxdt + \frac{\delta}{r^2} \int_{Q_{2r}\setminus Q_{r/2}} |u-\tilde{\tilde{u}}|^2 G_\theta^{1/2} dxdt$$

$$+\frac{c}{r^2}\int_{Q_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_\theta dxdt+\frac{c}{\delta r^2}\int_{Q_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_\theta^{3/2}dxdt$$

(2.22)

$$\leq \delta\int_{Q^\varepsilon_{2r}\setminus Q_{r/2}}|\nabla u|^2G_\theta^{1/2}dxdt+\delta\nu(\varepsilon)\int_{Q_{2r}\setminus Q_{r/2}}|\nabla u|^2G^{1/2}_{r^2}dxdt$$

$$+\frac{\delta}{r^2}\int_{Q^\varepsilon_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_\theta^{1/2}dxdt+\frac{\delta\nu(\varepsilon)}{r^2}\int_{Q_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G^{1/2}_{r^2}dxdt$$

$$+\frac{c}{r^2}\int_{Q^\varepsilon_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_\theta dxdt+\frac{c\nu(\varepsilon)}{r^2}\int_{Q_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_{r^2}dxdt$$

$$+\frac{c}{\delta r^2}\int_{Q^\varepsilon_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_\theta^{3/2}dxdt+\frac{c\nu(\varepsilon)}{\delta r^2}\int_{Q_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G^{3/2}_{r^2}dxdt.$$

Now choose $\delta = r^{-n/2}$. Via (2.17), (2.18) from (2.22) we derive

$$\Phi_\theta(\tfrac{r}{4})\leq c(\varepsilon)\int_{Q_{2r}\setminus Q_{r/2}}|\nabla u|^2G_\theta dxdt+\nu(\varepsilon)\int_{Q_{2r}}|\nabla u|^2G_{r^2}dxdt$$

$$+\frac{c(\varepsilon)}{r^2}\int_{Q^\varepsilon_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_\theta dxdt+\frac{c\nu(\varepsilon)}{r^2}\int_{Q_{2r}\setminus Q_{r/2}}|u-\tilde{\tilde{u}}|^2G_{r^2}dxdt.$$

Applying (2.17) again to draw the Green's function out of the last integrals, then using the Poincaré type inequality (2.6), and moving G_θ, G_{r^2} into the integrals again we finally obtain that

$$\Phi_\theta(\tfrac{r}{4})\leq c(\varepsilon)(\Phi_\theta(4r)-\Phi_\theta(\tfrac{r}{4}))+\nu(\varepsilon)\Phi_{r^2}(4r)$$

Choose ε such that $\nu(\varepsilon)<1$. Adding $c(\varepsilon)$ times the left to this inequality and dividing by $c(\varepsilon)+1$ there results

$$\Phi_\theta(\tfrac{r}{4})\leq\frac{c(\varepsilon)}{c(\varepsilon)+1}\Phi_\theta(4r)+\frac{\nu(\varepsilon)}{c(\varepsilon)+1}\Phi_{r^2}(4r)$$

$$\leq \mu\sup_{\theta>0}\Phi_\theta(4r)$$

with a constant $\mu = \frac{c(\varepsilon)+\nu(\varepsilon)}{c(\varepsilon)+1} < 1$. Now we also pass to the supremum with respect to $\theta > 0$ on the left hand side yielding

$$\sup_{\theta>0} \Phi_\theta(\tfrac{r}{4}) \leq \mu \sup_{\theta>0} \Phi_\theta(4r)$$

for all r such that $Q_{4r}(x_o,t_o) \subset Q_T$. Iterating, (2.6) implies that

$$\fint_{Q_r(x_o,t_o)} |u-\tilde{u}|^2 dxdt \leq cr^{-n} \int_{Q_{2r}(x_o,t_o)} |\nabla u|^2 dxdt$$

$$\leq c \int_{Q_{2r}(x_o,t_o)} |\nabla u|^2 G_{r^2} dxdt \leq c\, r^{2\alpha}$$

for some $\alpha > 0$, for all r such that $Q_{4r}(x_o,t_o) \subset Q_T$, with a uniform constant c depending only on the data and $\mathrm{dist}((x_o,t_o),S)$. But from Da Prato's result [5] again, we now infer Hölder continuity of u .

The preceding results may be strengthened to assert Hölder continuity of weak solutions of diagonal quasilinear systems under the assumption $a\,M < \lambda$ [12], which in genera is bestpossible.

3. The above results well confirm the impression that apart from technical complications all results of elliptic regularity theory have a parabolic analogue. This method of extrapolation, however, does not provide an answer for truly time dependent problems. For instance, does the parabolic "flow" conserve the regularity of the initial data, if only assumptions (0.3) , (0.4) are imposed and no smallnes condition is required? In this generality the question has found a negative answer [21], even for diagonal systems. However, if the elliptic system associated with the evolution problem has a variational structure and either $n = 2$ or a one-sided condition is satisfied, it is believed that regularity of initial data is retained.

Somewhat related is the problem of conveying the regularity theory for minima or quasi-minima of regular variational integrals to the associated parabolic problems. In the elliptic case the notion of quasi-minimum has proved a powerful and elegant tool (cp. [9]) and it would be highly desirable to make it available in the time-dependent case, perhaps by a time-discrete approximation of the evolution equations related with a functional in variation through a sequence of variational problems.

References

[1] Aronson, D.G.: Non-negative solutions of linear parabolic equations, Ann. Sc. Norm. Sup. Pisa 22(1968), 607-694.

[2] Biroli, M.: Existence of a Hölder continuous solution of a parabolic obstacle problem with quadratic growth nonlinearities, Boll. U.M.I.(6) 2-A(1983), 311-319.

[3] Campanato, S.: Equazioni ellittiche del secondo ordine e spazi $\mathfrak{L}^{2,\lambda}$, Ann. Mat. Pura Appl. 69(1965), 321-380.

[4] Campanato, S.: Equazioni paraboliche del secondo ordine e spazi $\mathfrak{L}^{2,\theta}(\Omega,\delta)$ e loro proprietà, Ann. Mat. Pura Appl. 73(1966), 55-102.

[5] Da Prato, G.: Spazi $L^{(p,\theta)}(\Omega,\delta)$ e loro proprietà, Ann. Mat. Pura Appl. 69(1965), 383-392.

[6] Gehring, F.W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping, Acta Math. 130 (1973), 265-277.

[7] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems, SFB-72 lecture notes 6, University of Bonn (1981).

[8] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems, Ann. Math. Studies 105, Princeton Univ. Press (1983).

[9] Giaquinta, M. and E. Giusti: Quasi-minima, Ann. Inst. H. Poincaré, Anal. Nonlin, (to appear 1984).

[10] Giaquinta, M. and S. Hildebrandt: Estimation á-priori des solutions faibles de certains systèmes non linéaires elliptiques, Sém. Goulanič-Meyer-Schwartz 198(1981), expos. 17.

[11] Giaquinta, M. and G. Modica: Regularity results for some classes of higher order elliptic systems, J. Reine Angew. Math. 311/312 (1979), 145-169.

[12] Giaquinta, M. and M. Struwe: An optimal regularity result for a class of quasilinear parabolic systems, manusc. math. 36(1981), 223-239.

[13] Giaquinta, M. and M. Struwe: On the partial regularity of weak solutions of nonlinear parabolic systems, Math. Z. 179(1982), 437-451.

[14] Hildebrandt, S. and K.O. Widman: Some regularity results for quasilinear elliptic systems of second order, Math. Z. 142(1975), 67-86.

[15] Hildebrandt S. and K.O. Widman: On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order, Ann. Sc. Norm. Sup. Pisa, Sér. 4, 4(1977), 145-178.

[16] Ladyshenskaya, O.A. , V.A. Solonnikov and N.N. Ural'ceva: Linear and quasilinear equations of parabolic type, Transl. Math. Monogr. 23, AMS, Providence, R.I. (1968).

[17] Ladyshenskaya, O.A. and N.N. Ural'ceva: Linear and quasilinear elliptic equations, Acad. Press, New York (1969).

[18] Morrey, C.B.: Multiple integrals in the calculus of variations, Springer, Berlin-Heidelberg-New York (1966).

19] Moser, J.: A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17(1964), 101-134, Errata, ibid. 20(1967), 231-236.

20] Struwe, M.: On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems, manusc. math. 35(1981), 125-145.

21] Struwe, M.: A counterexample in regularity theory for parabolic systems, Comm. Univ. Mat. Carol. (to appear).

22] Struwe, M. and M.A. Vivaldi: On the Hölder continuity of bounded weak solutions of quasi-linear parabolic inequalities, Ann. Mat. Pura Appl. (to appear)

23] Tolksdorf, P.: On some parabolic variational problems with quadratic growth, preprint.

24] Trudinger, N.S.: Pointwise estimates and quasilinear parabolic equations, Comm. Pure Appl. Math. 21(1968), 205-226.

25] Wiegner, M.: Ein optimaler Regularitätssatz für schwache Lösungen gewisser elliptischer Systeme, Math. Z. 147(1976), 21-28.

Institut für Angewandte Mathematik, Universität Bonn,
Beringstrasse 6, D-5300 Bonn 1, BRD

(Oblatum 25.5.1984)