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SOLVABILITY OF EVOLUTION PROBLEMS FOR VISCOUS INCOMPRESSIBLE FLOW IN DOMAINS WITH NON-COMPACT BOUNDARIES

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Abstract: One considers the question of solvability of initial-boundary value problems for the Stokes and Navier-Stokes equations in unbounded domains with non-compact boundaries assuming that the initial data and the external forces are not square integrable over the whole domain. For the linear problem the sketch of the proof of the existence theorem is given.

Key words: Stokes equation, Navier-Stokes equation, initial boundary value problems.

Classification: 35Q10, 76D05

We are concerned here with initial-boundary value problems for the Stokes and Navier-Stokes equations in domains $\Omega \subset \mathbb{R}^n$, $n = 2,3$, with several "exits to the infinity", i.e. in domains of the form

$$\Omega = \Omega_0 \cup G_1 \cup \dots \cup G_m, \quad \Omega_0' = \{x \in \Omega : |x| \leq R_0\}$$

where G_i, $i = 1,\dots,m$, are disjoint unbounded domains. It is assumed that for arbitrary $i = 1,\dots,m$, a sequence of bounded domains G_{ik}, $k = 1,2,\dots$ exists exhausting the "exit" G_i as $k \to \infty$ and possessing the following properties:

i) $G_{ik+1} \supset G_{ik}$

ii) The domains $\omega_{ik} = G_{ik} \setminus G_{ik-1}$ $(k=1,2,\dots,G_{io} = \emptyset)$

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as well as $\Omega_k = \Omega_0 \cup G_{1k} \cup \ldots \cup G_{mk}$ are connected and $\operatorname{dist}(\Omega \setminus \Omega_\ell, \Omega_0) \to \infty$ as $\ell \to \infty$.

(iii) Let $\vec{u}(x)$ be a divergence free vector field (i.e. $\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x_1} + \ldots + \frac{\partial u_n}{\partial x_n} = 0$) defined in G_j, locally square integrable with its first derivatives, vanishing on $\partial G_j \cap \partial \Omega$ and satisfying the condition $\int_{\Sigma_j} \vec{u} \cdot \vec{n}\, dS = 0$ where Σ_j is a section of G_j (for instance, $\Sigma_j = \partial \omega_{j\ell} \cap \partial \omega_{j\ell-1}$). For every $k \geqq 1$ a divergence free vector field $\vec{U}(x)$ exists such that $\vec{U}\big|_{\partial G_j \cap \partial \Omega} = 0$, $\vec{U}(x) = \vec{u}(x)$ for $x \in G_{jk-1}$, $\vec{U}(x) = 0$ for $x \in G_j \setminus G_{jk}$, the operator $P_{jk}: \vec{u} \to \vec{U}$ is linear and

$$\text{(1)} \qquad \|\vec{U}\|_{\omega_{jk}} \leq b\, \|\vec{u}\|_{\omega_{jk}}, \qquad \|\nabla \vec{U}\|_{\omega_{jk}} \leq b\, \|\nabla \vec{u}\|_{\omega_{jk}}$$

where $\|\vec{u}\|_{\omega_{jk}}$ is L_2-norm of $\vec{u}$ in ω_{jk}, $\nabla \vec{u} = \left\{ \frac{\partial u_i}{\partial x_j} \right\}_{i,j=1,\ldots,n}$ and b is a positive constant independent of $\vec{u}$, j, k.

The vector field $\vec{U} = P_{jk}\vec{u}$ satisfies the relations

$$\nabla \cdot \vec{U} = 0 \quad (x \in \omega_{jk}),$$

$$\text{(2)} \qquad \vec{U}\big|_{\partial \omega_{jk} \cap \partial \omega_{jk-1}} = \vec{u}\big|_{\partial \omega_{jk} \cap \partial \omega_{jk-1}}, \quad \vec{U}\big|_{\partial \omega_{jk} \setminus \partial \omega_{jk-1}} = 0$$

and can be represented in the form

$$\text{(3)} \qquad \vec{U} = \vec{u}\varsigma + \vec{v}$$

where ς is a smooth function, $0 \leqq \varsigma(x) \leqq 1$, $\varsigma(x) = 1$ for $x \in G_{jk-1}$, $\varsigma(x) = 0$ for $x \in G_j \setminus G_{jk}$, and

$$\text{(4)} \qquad \nabla \cdot \vec{v} = -\vec{u} \cdot \nabla \varsigma \equiv \varphi \quad (x \in \omega_{jk}), \quad \vec{v}\big|_{\partial \omega_{jk}} = 0$$

(as far as the problems (2),(4) are concerned, see [1 - 4]). The function φ satisfies the necessary condition $\int_{\omega_{jk}} \varphi\, dx = -\int_{\partial \omega_{jk} \cap \partial \omega_{jk-1}} \vec{u} \cdot \vec{n}\, dS = 0$.

Example 1. G_j is a cylinder: $x' = (x_1, x_2) \in G'$, $x_3 > x_3^0$. Let $G_{jk} = G' \times (x_3^0, x_3^0 + k]$, then $\omega_{jk} = G' \times (x_3^0 + k - 1, x_3^0 + k]$, and the estimates (1) are obvious.

Example 2. G_j is a cone: $\frac{x}{|x|} \in g$, $|x| > R_0$ (g is a domain on a unit sphere in $\mathbb{R}^n$). We define G_{jk} as $\{x \in G_j;\ R_0 < |x| \leq 2^k R_0\}$, so that $\omega_{jk} = \{x \in G_j: 2^{k-1} R_0 < |x| \leq 2^k R_0\}$. The function ζ in (3) can be chosen in such a way that $|\nabla \zeta| \leq C_1 R_0^{-1} 2^{-k}$. The problem (4) is solvable, and

$$\|\nabla \vec{v}\|_{\omega_{jk}} \leq C_2 \|\rho\|_{\omega_{jk}} \tag{5}$$

with a constant C_2 independent of k. Making use of the Friedrichs inequality

$$\|\vec{v}\|_{\omega_{jk}} \leq C_3 R_0 2^k \|\nabla \vec{v}\|_{\omega_{jk}} \quad (\vec{v}\,\big|_{\partial\omega_{jk} \cap \partial\Omega} = 0)$$

we easily obtain

$$\|\vec{U}\|_{\omega_{jk}} \leq \|\vec{u}\|_{\omega_{jk}} + \|\vec{v}\|_{\omega_{jk}} \leq (1 + C_3 C_2 C_1) \|\vec{u}\|_{\omega_{jk}},$$

$$\|\nabla \vec{U}\|_{\omega_{jk}} \leq \|\nabla \vec{v}\|_{\omega_{jk}} + \|\nabla \vec{u}\|_{\omega_{jk}} + C_1 R_0^{-1} 2^{-k} \|\vec{u}\|_{\omega_{jk}} \leq$$

$$\leq (C_1 + C_2 C_1) R_0^{-1} 2^{-k} \|\vec{u}\|_{\omega_{jk}} + \|\nabla \vec{u}\|_{\omega_{jk}} \leq$$

$$\leq [1 + C_1 C_3 (1 + C_2)] \|\nabla \vec{u}\|_{\omega_{jk}}.$$

Example 3. $G_j = \{x \in \mathbb{R}^3; 0 < x_3 < 1,\ |x'| = \sqrt{x_1^2 + x_2^2} > R_0\}$. We take $G_{jk} = \{x \in G_j: R_0 < |x'| \leq 2^k R_0\}$, $\omega_{jk} = \{x \in G_j: 2^{k-1} R_0 < |x'| \leq 2^k R_0\}$ and define $\zeta(x')$ in such a way that $|\nabla \zeta(x')| \leq C_1 R_0^{-1} 2^{-k}$. In the present case the estimate (5) is replaced by $\|\nabla \vec{v}\|_{\omega_{jk}} \leq C_4 2^k R_0 \|\rho\|_{\omega_{jk}}$. On the other hand, the constant in the Friedrichs inequality is independent of k, since

$$\|\vec{v}\|_{\omega_{jk}} \leqq \left\|\frac{\partial \vec{v}}{\partial x_3}\right\|_{\omega_{jk}}, \text{ if } \vec{v}\Big|_{\partial\omega_{jk}\cap\partial\Omega} = 0.$$

Hence, (1) follows.

The less elementary examples are presented in [3,4]. We now pass on to the definition of some functional spaces. We introduce the following notations:

Ω' : an arbitrary bounded subdomain of Ω . $W_2^{\ell}(\Omega')$: the S.L. Sobolev space consisting of vector fields that are square integrable in Ω' together with their generalized derivatives up to the order ℓ ; $\|\vec{u}\|_{W_2^{\ell}(\Omega')} = (\sum_{|a|\leqq\ell} \|D^a \vec{u}\|^2_{\Omega'})^{1/2}$.

q : the set of positive numbers q_k, $k = 0,1,\dots$ such that

$$q_{k+1} \geqq q_k, \quad q_{k+\ell} \leqq a_0 a_1^{\ell} q_k, \tag{6}$$

$\varkappa$: the set of positive numbers $\varkappa_{ik}$, $i = 1,\dots,m$, $k = 1,2,\dots$ such that

$$\begin{aligned} \varkappa_{ik+\ell} &\leqq \varkappa_{ik}\, a_0' a_1'^{\ell}, \quad \ell > 0, \\ \varkappa_{ik-\ell} &\leqq \varkappa_{ik}\, a_0' a_1'^{\ell}, \quad \ell = 0,\dots,k-1. \end{aligned} \tag{7}$$

The constants a_i, a_i' are positive and independent of k, ℓ; a_1, $a_1' > 1$.

$W_2^{\ell}(\Omega, q)$, $\mathcal{W}_2^{\ell}(\Omega, \varkappa)$: the spaces of vector fields $\vec{u} \in W^{\ell}_{2,loc}(\Omega)$ equipped with the norms

$$\|\vec{u}\|_{W_2^{\ell}(\Omega, q)} = (\sup_{k \geqq 0} q_k^{-1} \|\vec{u}\|^2_{W_2^{\ell}(\Omega_k)})^{1/2},$$

$$\|\vec{u}\|_{\mathcal{W}_2^{\ell}(\Omega,\varkappa)} = \Big[\max(\|\vec{u}\|^2_{W_2^{\ell}(\Omega_0)}, \sup_{i,k} \varkappa_{ik}^{-1} \|\vec{u}\|^2_{W_2^{\ell}(\omega_{ik})})\Big]^{1/2}.$$

For $\ell = 0$, we denote these spaces by $L_2(\Omega, q)$ and $\mathcal{L}_2(\Omega, \varkappa)$, respectively.

We observe that $\mathcal{W}_2^{\ell}(\Omega, \varkappa) \subset W_2^{\ell}(\Omega, q)$ with $q_k = 1 + \sum_{i=1}^{m} \sum_{j=1}^{k} \varkappa_{ij}$, since

$$\|\vec{u}\|^2_{W_2^\ell(\Omega_k)} \leq \|\vec{u}\|^2_{W_2^\ell(\Omega_0)} + \sum_{i=1}^{m}\sum_{j=1}^{k} \|\vec{u}\|^2_{W_2^\ell(\omega_{ij})} \leq$$
$$\leq (1 + \sum_{i=1}^{m}\sum_{j=1}^{k} \varkappa_{ij}) \|\vec{u}\|^2_{\mathcal{W}_2^\ell(\Omega,\varkappa)}.$$

Next, we introduce some spaces of vector fields depending both on x and on $t \in (0,T)$. Let $Q_T = \Omega \times (0,T)$, $Q_T' = \Omega' \times (0,T)$, $Q_{kT} = \Omega_k \times (0,T)$, $Q_T^{ij} = \omega_{ij} \times (0,T)$. By $L_2(Q_T, q)$ and $\mathcal{L}_2(Q_T, \varkappa)$ we mean the spaces of vector fields $\vec{u} \in L_{2,loc}(Q_T)$ with finite norms

$$\|\vec{u}\|_{L_2(Q_T,q)} = (\sup_k q_k^{-1} \|\vec{u}\|^2_{L_2(Q_{kT})})^{1/2}$$

and

$$\|\vec{u}\|_{\mathcal{L}_2(Q_T,\varkappa)} = \left[\max (\|\vec{u}\|^2_{L_2(Q_{0T})}, \sup_{i,k} \varkappa_{ik}^{-1} \|\vec{u}\|^2_{L_2(Q_T^{ik})})\right]^{1/2}$$

respectively. The following spaces play a basic role in subsequent considerations.

$D_0^{0,1/2}(Q_T')$: the space of vector fields with the norm given by the formula

$$\|\vec{u}\|^2_{D_0^{0,1/2}(Q_T')} = \int_{\Omega'} dx \int_{-\infty}^{T} dt \int_0^\infty |\vec{u}_0(x,t) - \vec{u}_0(x,t-h)|^2 \frac{dh}{h^2} =$$

$$(8) = \int_{\Omega'} dx \int_0^T dt \int_0^t |\vec{u}(x,t) - \vec{u}(x,t-h)|^2 \frac{dh}{h^2} +$$
$$+ \int_{\Omega'} dx \int_0^T |\vec{u}(x,t)|^2 \frac{dt}{t}$$

where $\vec{u}_0(x,t) = \vec{u}(x,t)$ for $t > 0$, $\vec{u}_0(x,t) = 0$ for $t < 0$.

$D_0^{1,1/2}(Q_T')$: the space of vector fields with the norm

$$(9) \quad \|\vec{u}\|_{D_0^{1,1/2}(Q_T')} = \left\{ \|\vec{u}\|^2_{D_0^{0,1/2}(Q_T')} + \right.$$
$$\left. + \int_0^T \int_{\Omega'} |\nabla \vec{u}(x,t)|^2 \, dxdt \right\}^{1/2}.$$

The boundedness of the norm (8) or (9) means that $\vec{u}(x,0) = 0$

in a certain sense. For $T = \infty$ these norms can be expressed in terms of the Fourier transform of $\vec{u}_0$ that is defined by $\vec{\tilde{u}}(x,\xi) = \int_0^\infty e^{-it\xi}\,\vec{u}_0(s,t)dt$, if the integral in the right-hand side is convergent. The norms (8) and (9) are equivalent to

$$\left(\int_{-\infty}^{\infty} d\xi \int_{\Omega'} |\xi|\, |\vec{\tilde{u}}(x,\xi)|^2\, dx\, d\xi \right)^{1/2} \text{ and}$$

$$\left[\int_{-\infty}^{\infty} d\xi \int_{\Omega'} (|\xi|\, |\vec{\tilde{u}}(x,\xi)|^2 + |\nabla \vec{\tilde{u}}|^2) dx\, d\xi \right]^{1/2} \text{ respectively.}$$

Moreover, it can be easily verified that for all $\sigma \geq 0$ the norm

$$\left[\int_0^\infty e^{-2\sigma t}\, dt \int_{\Omega'} dx \int_0^\infty |\vec{u}_0(x,t) - \vec{u}_0(x,t-h)|^2 \frac{dh}{h^2} + \right.$$

$$\left. + \int_0^\infty e^{-2\sigma t}\, dt \int_{\Omega'} (\sigma |\vec{u}(x,t)|^2 + |\nabla \vec{u}(x,t)|^2)\, dx \right]^{1/2}$$

is equivalent to

(10) $$\left[\int_{-\infty}^{\infty} d\xi \int_{\Omega'} (|s|\, |\vec{\tilde{u}}(x,s)|^2 + |\nabla \vec{\tilde{u}}(x,s)|^2)\, dx \right]^{1/2}$$

where $s = \sigma + i\xi$ and $\vec{\tilde{u}}(x,s) = \int_0^\infty e^{-st}\,\vec{u}(x,t)dt$ is the Laplace transform of $\vec{u}$ with respect to t.

Following the above scheme, we define $D_0^{0,1/2}(Q_T,q)$ and $\mathcal{D}_0^{0,1/2}(Q_T,\varkappa)$ as the spaces of vector fields equipped with the norms

$$\|\vec{u}\|_{D_0^{0,1/2}(Q_T,q)} = \left[\sup_k q_k^{-1} \|\vec{u}\|^2_{D_0^{0,1/2}(Q_{kT})} \right]^{1/2}$$

and

$$\|\vec{u}\|_{\mathcal{D}_0^{0,1/2}(Q_T,\varkappa)} = \left[\max \left(\|u\|^2_{D_0^{0,1/2}(Q_{0T})}, \right. \right.$$

$$\left. \left. \sup \varkappa_{ik}^{-1} \|\vec{u}\|^2_{D_0^{0,1/2}(Q_T^{ik})} \right) \right]^{1/2}$$

respectively. The spaces $D_0^{1,1/2}(Q_T,q)$ and $\mathcal{D}_0^{1,1/2}(Q_T,\varkappa)$ are defined in an analogous way.

Let $\vec{\varphi} \in L_2(\Omega,q)$. We say that $\vec{u} \in D_0^{0,1/2}(Q_T,q)$ if

$\vec{u}(x,t) = \vec{\varphi}(x)\,\Phi(t) + \vec{v}(x,t)$ where $\vec{v} \in D_0^{0,1/2}(Q_T, q)$ and $\Phi(t)$ is a fixed smooth function such that $0 \leqq \Phi(t) \leqq 1$, $\Phi(t) = 0$ for $t \geqq 1$, $\Phi(t) = 1$ for $0 \leqq t \leqq 1/2$, and we set

$$\|\vec{u}\|_{D_{\vec{\varphi}}^{0,1/2}(Q_T,q)} = \left(\|\vec{v}\|^2_{D_0^{0,1/2}(Q_T,q)} + \|\vec{\varphi}\|^2_{L_2(\Omega,q)}\right)^{1/2}.$$

This expression is equivalent to the norm

$$\int_{-\infty}^{\infty} dt \int_\Omega dx \int_0^\infty |\vec{u}_{\vec{\varphi}}(x,t) - \vec{u}_{\vec{\varphi}}(x,t-h)|^2 \frac{dh}{h^2})^{1/2} \quad \text{where}$$

$\vec{u}_{\vec{\varphi}}(x,t) = \vec{u}(x,t)$ for $t > 0$ and $\vec{u}_{\vec{\varphi}}(x,t) = \vec{\varphi}(x)\,\Phi(-t)$ for $t < 0$. If $\vec{\varphi} \in W_2^1(\Omega, q)$, then $D_{\vec{\varphi}}^{1,1/2}(Q_T, q)$ is the set of all $\vec{u} = \vec{\varphi}\,\Phi + \vec{v}$, $\vec{v} \in D_0^{1,1/2}(Q_T, q)$, and

$$\|\vec{u}\|_{D_{\vec{\varphi}}^{1,1/2}(Q_T,q)} = \left(\|\vec{v}\|^2_{D_0^{1,1/2}(Q_T,q)} + \|\vec{\varphi}\|^2_{W_2^1(\Omega,q)}\right)^{1/2}.$$

The sets $\mathcal{D}_{\vec{\varphi}}^{0,1/2}(Q_T, \varkappa)$ and $\mathcal{D}_{\vec{\varphi}}^{1,1/2}(Q_T, \varkappa)$ are defined for $\vec{\varphi} \in \mathcal{L}_2(\Omega, \varkappa)$ and $\vec{\varphi} \in \mathcal{W}_2^1(\Omega, \varkappa)$ as the sets of $\vec{u} = \vec{\varphi}(x)\,\Phi(t) + \vec{v}$ with $\vec{v} \in \mathcal{D}_0^{0,1/2}(Q_T, \varkappa)$ or $\vec{v} \in \mathcal{D}_0^{1,1/2}(Q_T, \varkappa)$ respectively, and

$$\|\vec{u}\|_{\mathcal{D}_{\vec{\varphi}}^{0,1/2}(Q_T,\varkappa)} = \left(\|\vec{v}\|^2_{\mathcal{D}_0^{0,1/2}(Q_T,\varkappa)} + \|\vec{\varphi}\|^2_{\mathcal{L}_2(\Omega,\varkappa)}\right)^{1/2},$$

$$\|u\|_{\mathcal{D}_{\vec{\varphi}}^{1,1/2}(Q_T,\varkappa)} = \left(\|\vec{v}\|^2_{\mathcal{D}_0^{1,1/2}(Q_T,\varkappa)} + \|\vec{\varphi}\|^2_{\mathcal{W}_2^1(\Omega,\varkappa)}\right)^{1/2}.$$

We now turn to the initial-boundary value problem

$$\frac{\partial \vec{v}}{\partial t} - \nabla^2 \vec{v} + \nabla p = \vec{f} - \sum_{j=1}^{m} \frac{\partial \vec{f}_j}{\partial x_j}, \quad \nabla \cdot \vec{v} = 0, \ (x,t) \in Q_T,$$

(11)

$$\vec{v}|_{t=0} = \vec{\varphi}(x), \quad \vec{v}|_{x \in \partial\Omega} = 0,$$

(12) $\displaystyle\int_{\Sigma_j} \vec{v} \cdot \vec{n}\, dS = \alpha_j(t), \quad j = 1, \dots, m-1.$

This problem as well as a similar problem for the full Navier-Stokes equations is studied in the book [5] by O.A. Ladyzhenskaya both for bounded and for unbounded domains. The solutions are found in classes of vector fields whose elements have a finite "energy integral" (so that in particular $\nabla \vec{v} \in L_2(Q_T)$) and satisfy the homogeneous conditions (12), but these conditions are not written explicitly. It was J. Heywood who introduced the conditions (12) into the formulation of the problem and who found the solutions of (11),(12) with arbitrary $\alpha_j(t)$ in a rather particular class of domains [6]. This class was considerably extended in [7 - 9, 3, 1].

In the present paper we study the problem (11),(12) in a generalized (i.e. weak) formulation, but unlike [5 - 9, 1], we do not require the boundedness of the "energy integral". For the case of cylindrical $\Omega = \omega \times \mathbb{R}$ the linear and non-linear evolution problems of viscous flow are studied in a certain class of vector fields with an infinite "energy integral" by O.A. Ladyzhenskaya, H. True and the present author [10]. In the paper [11] devoted to a linearized evolution problem, the class $D^{1,1/2}$ is used.

By a weak solution of (11),(12) we mean a divergence free vector field $\vec{v}(x,t)$ that is locally square integrable in Q_T, as well as its first derivatives $\vec{v}_{x_i}$ and that satisfies the conditions (12), the homogeneous boundary conditions $v|_{x\in\partial\Omega} = 0$ and the integral identity

$$(13)\qquad \int_0^T \int_\Omega (-\vec{v}\cdot\vec{\eta}_t + \nu\nabla\vec{v} : \nabla\vec{\eta})\,dxdt = \int_0^T \int_\Omega (\vec{f}\cdot\vec{\eta} + \sum_{j=1}^{n} \vec{f}_j\cdot\vec{\eta}_{x_j})\,dxdt + \int_\Omega \vec{\varphi}(x)\cdot\vec{\eta}(x,0)\,dx$$

where $\vec{\varphi}\cdot\vec{\eta} = \varphi_1\eta_1 + \ldots + \varphi_n\eta_n$, $\nabla\vec{v} : \nabla\vec{\eta} = \sum_{i,j=1}^{n} \frac{\partial v_i}{\partial x_j}\frac{\partial \eta_i}{\partial x_j}$

and $\vec{\eta}$ is an arbitrary divergence free vector field with a compact support, possessing the derivatives $\vec{\eta}_t$, $\vec{\eta}_{x_i} \in L_2(Q_T)$ and vanishing for $x \in \partial\Omega$ and for $t = T$.

Theorem 1. Suppose that $T < \infty$ and that

i) $\vec{\varphi} \in \mathcal{W}_2^1(\Omega, \varkappa)$, is a divergence free vector field,

$$\vec{\varphi}\,\big|_{\partial\Omega} = 0,$$

ii) $\vec{f} \in \mathcal{L}_2(Q_T, \varkappa)$, $\vec{f}_i \in \mathcal{L}_2(Q_T, \varkappa)$,

iii) there exists a divergence free vector field $\vec{a} \in \mathcal{D}_o^{1,1/2}(Q_T, \varkappa)$ vanishing for $x \in \partial\Omega$ and satisfying the conditions

$$\int_{\Sigma_j} \vec{a}\cdot\vec{n}\, ds = \alpha_j(t) - \Phi(t) \int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}\, dS$$

(it means in particular that the compatibility condition $\alpha_j(0) = \int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}\, dS$ holds).

If the constant a_1' in (7) satisfies the condition $1 < a_1' < 1 + \delta$, then the problem (11),(12) has a unique weak solution $\vec{v} \in \mathcal{D}_{\vec{\varphi}}^{1,1/2}(Q_T, \varkappa)$ and

$$\|\vec{v}\|_{\mathcal{D}_{\vec{\varphi}}^{1,1/2}(Q_T,\varkappa)} \leq C(T)\Big(\|\vec{f}\|_{\mathcal{L}_2(Q_T,\varkappa)} + \sum_{j=1}^{m} \|\vec{f}_j\|_{\mathcal{L}_2(Q_T,\varkappa)} + \tag{14}$$
$$+ \|\vec{\varphi}\|_{\mathcal{W}_2^1(\Omega,\varkappa)} + \|\vec{a}\|_{\mathcal{D}_o^{1,1/2}(Q_T,\varkappa)}\Big) \equiv C(T)\, M_1(T).$$

If, in addition, $\vec{f}_j \in \mathcal{D}_o^{0,1/2}(Q_T, \varkappa)$, $\vec{\varphi} \in \mathcal{W}_2^2(\Omega, \varkappa)$, $\vec{a}_t \in \mathcal{L}_2(Q_T, \varkappa)$, $\nabla\vec{a} \in \mathcal{D}_o^{0,1/2}(Q_T, \varkappa)$, then $\vec{v}_t \in \mathcal{L}_2(Q_T, \varkappa)$, $\nabla\vec{v} \in \mathcal{D}_{\nabla\vec{\varphi}}^{0,1/2}(Q_T, \varkappa)$, and

$$\|\vec{v}_t\|_{\mathcal{L}_2(Q_T,\varkappa)} + \|\nabla\vec{v}\|_{\mathcal{D}_{\nabla\vec{\varphi}}^{0,1/2}(Q_T,\varkappa)} \leq C_1(T)\Big(\|\vec{f}\|_{\mathcal{L}_2(Q_T,\varkappa)} +$$
$$+ \sum_{j=1}^{m} \|\vec{f}_j\|_{\mathcal{D}_o^{0,1/2}(Q_T,\varkappa)} + \|\vec{\varphi}\|_{\mathcal{W}_2^2(Q,\varkappa)} + \|\vec{a}_t\|_{\mathcal{L}_2(Q_T,\varkappa)} +$$
$$\|\nabla\vec{a}\|_{\mathcal{D}_o^{0,1/2}(Q_T,\varkappa)}\Big) \equiv C_1(T)\, M_2(T). \tag{15}$$

An analogous existence theorem holds for the spaces $D^{1,1/2}_{\vec{\varphi}}(Q_T, q)$. It is formulated in the same way as Theorem 1 with obvious changes; the conditions $\vec{\varphi} \in \mathcal{W}^1_2(\Omega, \varkappa)$, $\vec{f} \in \mathcal{L}_2(Q_T, \varkappa)$ should be replaced by $\vec{\varphi} \in W^1_2(\Omega, q)$, $\vec{f} \in L_2(Q_T, q)$ etc. The constant a_1 in (6) should not be too large, i.e. $1 < a_1 < 1 + \delta_1$, $\delta_1 > 0$.

Consider the non-linear problem

$$
(16)\quad
\begin{aligned}
&\vec{v}_t + (\vec{v}\cdot\nabla)\vec{v} - \nu\nabla^2\vec{v} + \nabla p = \vec{f}, \quad \nabla\cdot\vec{v} = 0\\
&\vec{v}|_{t=0} = \vec{\varphi}(x), \quad \vec{v}|_{x\in\partial\Omega} = 0,
\end{aligned}
$$

$$
(17)\quad \int_{\Sigma_j} \vec{v}\cdot\vec{n}\, dS = \alpha_j(t), \quad j = 1,\dots,m-1.
$$

A weak solution of (16),(17) is defined as a divergence free vector field $\vec{v}(x,t) \in L_{2,loc}(Q_T)$ with $\nabla\vec{v} \in L_{2,loc}(Q_T)$, vanishing for $x \in \partial\Omega$ and satisfying (17) and the integral identity

$$
\int_0^T\int_\Omega (-\vec{v}\cdot\vec{\eta}_t - (\vec{v}\cdot\nabla)\vec{\eta}\cdot\vec{v} + \nu\,\nabla\vec{v}\,\nabla\vec{\eta}\,)\,dx\,dt =
$$
$$
= \int_0^T\int_\Omega \vec{f}\cdot\vec{\eta}\,dx\,dt + \int_\Omega \vec{\varphi}\cdot\vec{\eta}\,dx
$$

for any divergence free $\vec{\eta}$ with a compact support such that $\vec{\eta}$, $\vec{\eta}_t$, $\nabla\vec{\eta} \in L_2(Q_T)$, $\vec{\eta} = 0$ for $x \in \partial\Omega$ and for $t = T$.

Theorem 2. Let the domain Ω satisfy the following additional condition: for every $\vec{w}(x)$ specified in ω_{jk} possessing the finite Dirichlet integral and vanishing for $x \in \partial\omega_{ij} \cap \cap \partial\Omega$, the inequalities

$$
(18)\quad
\begin{aligned}
&\|\vec{w}\|_{L_6(\omega_{ij})} \leqq b\,\|\nabla\vec{w}\|_{\omega_{ij}}, \quad \text{if } n = 3\\
&\|\vec{w}\|_{L_6(\omega_{ij})} \leqq b\,\|\nabla\vec{w}\|^{1/3}_{\omega_{ij}}\ \|\vec{w}\|^{2/3}_{\omega_{ij}}, \quad \text{if } n = 2
\end{aligned}
$$

hold with a constant b independent of $\vec{w}$ and j, and let the same inequalities hold for the domain Ω_0. If $\vec{f}, \vec{\varphi}, \vec{a}$ satisfy all the hypotheses of Theorem 1 and $\varkappa_{ij} \leq 1$, then the problem (16), (17) has a unique solution $\vec{v} \in \mathcal{D}^{1,1/2}_{\vec{\varphi}}(Q_T, \varkappa)$ with $\vec{v}_t \in \mathcal{L}_2(Q_{T_1}, \varkappa)$, $\nabla \vec{v} \in \mathcal{D}^{0,1/2}_{\nabla\vec{\varphi}}(Q_{T_1}, \varkappa)$ in the interval $(0, T_1)$, T_1 being a non-increasing function of $M_1(T) + M_2(T)$ ($M_i(T)$ are the same as in (14),(15)).

The proofs of Theorems 1 and 2 and of analogous theorems in other functional spaces are given in [4]. We restrict ourselves to the idea of the proof of the first part of Theorem 1. First of all, we construct a weak solution of the problem (11),(12) $\vec{v} \in D^{1,1/2}_{\vec{\varphi}}(Q_T, q)$ with $q_n = 1 + \sum_{i=1}^{m} \sum_{j=1}^{m} \varkappa_{ij}$. We extend the vector fields $\vec{f}, \vec{f}_i, \vec{a}$ into the half-space $t < 0$ by zero and then into the half-space $t > T$ as even functions of $t - T$. This extension conserves the differentiability properties of $\vec{f}, \vec{f}_i, \vec{a}$. The new unknown vector field $\vec{u} = \vec{v} - \Phi(t)\vec{\varphi}(x) - \vec{a}(x,t)$ is divergence free, it satisfies the conditions $\vec{u}|_{x \in \partial\Omega} = 0$, $\int_{\Sigma_j} \vec{u} \cdot \vec{n}\, dS = 0$, $j = 1, \dots, m$, and the integral identity

$$(19)\qquad \begin{aligned} \int_0^\infty \int_\Omega (-\vec{u} \cdot \vec{\eta}_t + \nu \nabla \vec{u} : \nabla \vec{\eta})\, dx dt &= \int_0^\infty \int_\Omega (\vec{f} \cdot \vec{\eta} + \sum_i \vec{f}_i \cdot \vec{\eta}_{x_i})\, dx dt - \\ &- \int_0^\infty \int_\Omega (\Phi_t \vec{\varphi} \cdot \vec{\eta} - \vec{a} \cdot \vec{\eta}_t)\, dx dt - \nu \int_0^\infty \int_\Omega (\Phi(t) \nabla \vec{\varphi} : \nabla \vec{\eta} + \\ &+ \nabla \vec{a} : \nabla \vec{\eta})\, dx dt \end{aligned}$$

that is a consequence of the identity (13) written for $T = \infty$. Changing $\vec{\eta}$ for $\vec{\eta}\, e^{-2\sigma t}$ and applying the Parceval's identity, we can rewrite (19) in the form

$$(20)\qquad \begin{aligned} &\int_{-\infty}^{\infty} d\xi \int_\Omega (s\, \vec{\tilde{u}} \cdot \vec{\tilde{\eta}} + \nu \nabla \vec{\tilde{u}} : \nabla \vec{\tilde{\eta}})\, dx = \\ &= \int_{-\infty}^{\infty} d\xi \int_\Omega (\vec{\tilde{g}} \cdot \vec{\tilde{\eta}} + \sum_i \vec{\tilde{g}}_i \cdot \vec{\tilde{\eta}}_{x_i} + s \vec{\tilde{g}}_0 \cdot \vec{\tilde{\eta}})\, dx \end{aligned}$$

where $\vec{\tilde{u}} \cdot \vec{\bar{\eta}} = \tilde{u}_1 \bar{\tilde{\eta}}_1 + \ldots + \tilde{u}_n \bar{\tilde{\eta}}_n$, $\vec{\tilde{u}}$ is the Laplace transform of $\vec{u}$, $\vec{g} = \vec{f} - (\tilde{\Phi}_t)\vec{\varphi}(x)$, $\vec{g}_o = -\vec{a}$, $\vec{g}_i = \vec{f}_i - \nu \tilde{\Phi} \frac{\partial \varphi}{\partial x_i} - \nu \frac{\partial \vec{a}}{\partial x_i}$.

Consider the following auxiliary problem:

Find a divergence free vector field $\vec{\tilde{u}}^{(\ell)} \in \overset{\circ}{W}{}^1_2(\Omega_\ell)$ depending on a parameter $s = \sigma + i\xi$ and satisfying the integral identity

$$Q(\vec{\tilde{u}}^{(\ell)}, \vec{\psi}) = \int_\Omega (s\vec{\tilde{u}}^{(\ell)} \cdot \vec{\psi} + \nu \nabla \vec{\tilde{u}}^{(\ell)} : \nabla \vec{\psi})\, dx =$$

$$(21) \qquad = \int_{\Omega_\ell} [(\vec{g} + s\vec{g}_o) \cdot \vec{\psi} + \sum_i \vec{g}_i \cdot \vec{\psi}_{x_i}]\, dx$$

for all divergence free $\vec{\psi} \in \overset{\circ}{W}{}^1_2(\Omega_\ell)$. For $\sigma \geq 0$ the quadratic form $Q(\vec{w}, \vec{w})$ satisfies the condition

$$\mathrm{Re}(1 - i \operatorname{sgn} \xi) Q(\vec{w}, \vec{w}) \geq (\sigma + |\xi|) \int_{\Omega_\ell} |\vec{w}|^2\, dx +$$

$$(22) \qquad + \nu \int_{\Omega_\ell} |\nabla \vec{w}|^2\, dx \geq \int_{\Omega_\ell} (|s|\, |\vec{w}|^2 + \nu\, |\nabla \vec{w}|^2)\, dx,$$

and the existence of $\vec{\tilde{u}}^{(\ell)}$ follows from the Lax-Milgram theorem. Moreover, $\vec{\tilde{u}}^{(\ell)}$ is a holomorphic function of s in the half-plane $\mathrm{Re}\, s > 0$, since $\vec{g} + s\vec{g}_o$ and $\vec{g}_i$ are holomorphic. Letting $\vec{\psi} = \vec{\tilde{u}}^{(\ell)} (1 + i \operatorname{sgn} \xi)$ and integrating with respect to ξ, we obtain after elementary calculations the estimate:

$$\int_{-\infty}^{\infty} d\xi \int_{\Omega_\ell} (|s|\, |\vec{\tilde{u}}^{(\ell)}|^2 + \nu\, |\nabla \vec{\tilde{u}}^{(\ell)}|^2)\, dx \leq$$

$$(23) \quad \leq \int_{-\infty}^{\infty} d\xi \int_{\Omega_\ell} \left(\frac{c_2}{\sigma} |\vec{g}|^2 + c_3 |\vec{g}_o|^2 |s| + c_3 \sum_{i=1}^{m} |\vec{g}_i|^2\right) dx \leq$$

$$\leq c_4(\sigma)\, M_1 q_\ell \, .$$

Next, we evaluate $\int_{-\infty}^{\infty} d\xi \int_{\Omega_k} (|s|\, |\vec{\tilde{u}}^{(\ell)}|^2 + \nu\, |\nabla \vec{\tilde{u}}^{(\ell)}|^2)\, dx$ for $k < \ell$. To this end we insert into (21) the test function $\vec{\psi} = \vec{U}_{k+1} (1 + i \operatorname{sgn} \xi)$ where $\vec{U}_{k+1} = \vec{\tilde{u}}^{(\ell)}$ in Ω_k, $\vec{U}_{k+1} = 0$ in $\Omega_\ell \setminus \Omega_{k+1}$, $\vec{U}_{k+1} = P_{ik+1} \vec{\tilde{u}}^{(\ell)}$ in ω_{ik+1}. After easy calculations, making use of (1), we obtain

$$\int_{-\infty}^{\infty} d\xi \int_{\Omega_k} (|s||\vec{u}^{(\ell)}|^2 + \nu |\nabla \vec{u}^{(\ell)}|^2)dx \leqq$$

$$(24) \quad \leqq C_5 \int_{-\infty}^{\infty} d\xi \int_{\Omega_{k+1}\setminus\Omega_k} (|s||\vec{u}^{(\ell)}|^2 + \nu |\nabla \vec{u}^{(\ell)}|^2)dx +$$

$$+ C_6 \int_{-\infty}^{\infty} d\xi \int_{\Omega_{k+1}} \left(\frac{1}{\sigma}|\vec{g}|^2 + |s||\vec{g}_0|^2 + \sum_i |\vec{g}_i|^2\right) dx,$$

i.e.

$$y_k \leqq C_5(y_{k+1} - y_k) + C_7 M_1 q_k$$

or

$$y_k \leqq \frac{C_5}{C_5+1} y_{k+1} + \frac{C_7}{C_5+1} M_1 q_k$$

where y_k is the left-hand side of (24). Hence,

$$y_k \leqq \frac{C_7 M_1}{C_5+1}\left\{ q_k + \frac{C_5}{C_5+1} q_{k+1} + \dots + \left(\frac{C_5}{C_5+1}\right)^{\ell-k-1} q_{\ell-1}\right\} +$$

$$(25) \quad + \left(\frac{C_5}{C_5+1}\right)^{\ell-k} y_\ell \leqq \frac{C_7 M_1}{C_5+1} q_k \left\{ 1 + a_0 \frac{C_5 a_1}{C_5+1} + \dots +\right.$$

$$\left. + a_0 \left(\frac{C_5 a_1}{C_5+1}\right)^{\ell-k-1}\right\} + C(\sigma) M_1 \left(\frac{C_5 a_1}{C_5+1}\right)^{\ell-k} q_k \leqq C_8 M_1 q_k, \quad \forall k < \ell,$$

provided $C_5 a_1 / C_5+1 < 1$.

In virtue of (25), there exists a subsequence of $\{\vec{u}^{(\ell)}\}$ converging weakly in the Hilbert spaces with norms (10) for arbitrary bounded $\Omega' \subset \Omega$. The limiting element $\vec{u}$ satisfies the inequality (25), i.e.

$$\int_{-\infty}^{\infty} d\xi \int_{\Omega_k} (|s||\vec{u}|^2 + \nu |\nabla \vec{u}|^2) dx \leqq C_8 M_1 q_k.$$

Moreover, making use of (21) it is not hard to show that $\vec{u}$ satisfies the identity (20). The inverse Laplace transform $\vec{u}_0$ of $\vec{u}$ vanishes for $t < 0$ (since $\vec{u}$ is a holomorphic function of s in the half-plane Re $s \geqq 0$), and satisfies the inequality

$$\int_0^\infty \frac{dh}{h^2} \int_{-\infty}^\infty e^{-2\sigma t}\, dt \int_{\Omega_k} |\vec{u}_0(x,t) - \vec{u}_0(x,t-h)|^2\, dx +$$

$$+ \int_0^\infty e^{-2\sigma t}\, dt \int_{\Omega_k} |\nabla \vec{u}|^2\, dx \leqq C_9\, M_1\, q_k$$

and the integral identity (19).

Let us show that $\vec{u} \in \mathfrak{D}_0^{1,1/2}(Q_T, \varkappa)$. We fix ω_{ik}, $k \geqq 1$ and take $\vec{\eta} = (1 + i \operatorname{sgn} \xi) [\vec{\vartheta}_{k+p+1} - \vec{\vartheta}_{k-p-1}]$ in (20). Repeating the calculations leading to (24) we arrive at the following inequality for the numbers

$$z_p = \int_{-\infty}^\infty d\xi \int_{G_{ik+p} \setminus G_{ik-p-1}} (|s|\, |\vec{u}|^2 + \nu\, |\nabla \vec{u}|^2)\, dx, \quad p = 0,\dots,k-1:$$

$$z_p \leqq \frac{C_{10}}{C_{10}+1}\, z_{p+1} + C_{11} \int_{-\infty}^\infty d\xi \int_{G_{ik+p+1} \setminus G_{ik-p-2}} (\tfrac{1}{6} |\vec{g}|^2 + |s|\, |\vec{g}_0|^2 +$$

$$+ \sum_j |\vec{g}_j|^2)\, dx \leqq \frac{C_{10}}{C_{10}+1}\, z_{p+1} + C_{12}\, \tilde{q}_p\, M_1$$

where $\tilde{q}_p = \sum_{j=1}^{p+1} (\varkappa_{ik+j} + \varkappa_{ik-j}) + \varkappa_{ik}$ satisfy (6):

$$\tilde{q}_{p+t} = \tilde{q}_p + \sum_{j=p+2}^{p+t+1} (\varkappa_{ik+j} + \varkappa_{ik-j}) \leqq$$

$$\leqq \tilde{q}_p + a_0' (\varkappa_{ik+p+1} + \varkappa_{ik-p-1}) \sum_{j=p+2}^{p+t+1} a_1'^{\,j-p-1} \leqq a_0''\, \tilde{q}_p\, a_1'^{\,t}.$$

Hence,

$$z_1 \leqq C_{13}\, M_1 \tilde{q}_1 \leqq C_{14}\, M_1\, \varkappa_{ik}$$

provided $a_1' \leqq \dfrac{C_{10}+1}{C_{10}}$. This inequality shows that $\vec{u} \in \mathfrak{D}_0^{1,1/2}(Q_T, \varkappa)$, and the estimate (14) for $\vec{v} = \vec{u} + \vec{a} + \varphi \Phi$ follows immediately with $C(T) = Ce^{6T}$.

The second part of the theorem can be proved by similar estimates of the integrals

$$\int_{-\infty}^{\infty} d\xi \int_{\Omega_k} (|s|^2 |\vec{u}|^2 + \nu |s| |\nabla \vec{u}|^2)\, dx \text{ and}$$

$$\int_{-\infty}^{\infty} d\xi \int_{\omega_{ik}} (|s|^2 |\vec{u}|^2 + \nu |s| |\vec{u}|^2)\, dx.$$

With Theorem 1 established, it is not hard to prove the solvability of (16),(17) by successive approximations. Theorem 1 holds also for $T = \infty$ under slightly more restrictive assumptions on $\vec{f}$ and $\vec{\varphi}$ (see [4]), and if the data $\vec{f}$, $\vec{\varphi}$, $\vec{a}$ are small in an appropriate sense, then the problem (16),(17) has a solution for all $t > 0$. This is completely analogous to the result of O.A. Ladyzhenskaya [5] obtained in the three-dimensional case for flows with a finite "energy integral". The question, whether there exists the global solution $\vec{v} \in \mathfrak{D}^{1,1/2}_{\vec{\varphi}}(Q_T, \varkappa)$ of the problem (16),(17) in the two-dimensional case is still open.

A different approach to the problems (11),(12) and (16),(17) is proposed by M.E. Bogovski [12] who proved coercive estimates in the spaces $W_p^{2,1}(Q_T)$ for the solutions of (11),(12). This is a generalization of the present author's results for "interior" and "exterior" domains with a compact boundary [13, 14]. On the base of these estimates, the solvability of a non-linear problem is established locally for $n = 3$ and globally for $n = 2$. For p large enough, the space $W_p^{2,1}(Q_T)$ contains vector fields with an infinite "energy integral", but in some important cases, for instance, in the case of cylindrical Ω and $\alpha = \int_\Sigma \vec{v} \cdot \vec{n}\, dS \neq 0$, the solutions of (11),(12) and (16),(17) do not vanish at the infinity and do not belong to $W_p^{2,1}(Q_T)$.

The exterior problems are considered also in the Hölder spaces $C^{2+\alpha, 1+\alpha/2}(Q_T)$ without any assumption of the stabilization of the solution as $|x| \to \infty$. For the problem (11), for instance, the following theorem is established in [14].

Theorem 3. Let Ω be an exterior domain with $\partial\Omega \subset C^{2+\alpha}$, $\alpha \in (0,1)$, and suppose that $\vec{\varphi} \in C^{2+\alpha}(\Omega)$ is a divergence free vector field vanishing on $\partial\Omega$, $\vec{f} = 0$, $\vec{f}_j \in C^{1+\alpha,(1+\alpha)/2}(Q_T)$ and the compatibility condition holds

$$P_J\Big(-\sum_{j=1}^{m} \frac{\partial f_j(x,0)}{\partial x_j} + \nu \nabla^2 \vec{\varphi}(x)\Big)\Big|_{x\in\partial\Omega} = 0$$

where $P_J \vec{H} = \vec{H} - \nabla\omega$ and ω is a solution of the Neumann problem

$$\nabla^2\omega = \nabla \cdot \vec{H}, \quad \frac{\partial\omega}{\partial n}\Big|_{\partial\Omega} = \vec{H}\cdot n\Big|_{\partial\Omega} .$$

Then the problem (11) has a unique solution $v \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$ and, in addition, $p(x,t)$ possesses the finite norm

$$|p|^{(\alpha,\gamma)} = \sup_{x,y,t,\tau} |t-\tau|^{\frac{-(1+\alpha-\gamma)}{2}} |x-y|^{-\gamma} |p(x,t) - p(y,t) -$$

$- p(x,\tau) + p(y,\tau)|$, $\gamma \in (0,1)$. For the solution a coercive estimate holds

$$|\vec{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} + |p|^{(\alpha,\gamma)} \leqq$$

$$\leqq C(T) \Big(\sum_j |\vec{f}_j|_{C^{1+\alpha,(1+\alpha)/2}(Q_T)} + |\vec{\varphi}|_{C^{2+\alpha}(\Omega)} \Big).$$

The restrictions on $\vec{f}$, $\vec{f}_j$ can be weakened, but it should be pointed out that the formulation of this theorem given in [14] (see Theorem 9.1) needs some corrections. For the nonlinear problem (16) an analogous local theorem is established.

Because of the conditions (18), Theorem 2 does not seem to be applicable to exterior domains, and it would be interesting to find an appropriate generalization of this theorem. In this connection it should be noted that for linear parabolic second order equations some more sharp estimates are found, which makes it possible to work in the class of weak solutions whose "energy integrals" in the domains $\Omega_r = \{x \in \Omega : |x| < r\}$ may grow as fast as e^{ar^2} for $r \gg 1$ (see [15, 16]).

References

[1] ЛАДЫЖЕНСКАЯ О.А.; СОЛОННИКОВ В,А,: О некоторых задачах векторного анализа и об обобщенных постановках краевых задач для уравнений Навье-Стокса, Записки научных семинаров ЛОМИ, 59(1976), 81-116.

[2] БОГОВСКИЙ М.Е.: Решение некоторых задач векторного анализа, связанных с операторами div и grad , Труды семинара С.Л. Соболева 1(1980), 5-40.

[3] КАПИТАНСКИЙ Л.В., ПИЛЕЦКАС К.И.: О пространствах соленоидальных векторов в областях с некомпактными границами сложной формы. Препринт ЛОМИ Р-2-81, 1981.

[4] SOLONNIKOV V.A.: Stokes and Navier-Stokes equations in domains with non-compact boundaries, Nonlinear partial differential equations and their applications, Collège de France Seminar. Edited by H. Brézis & J.L. Lions, Volume IV, 1083.

[5] ЛАДЫЖЕНСКАЯ О.А.: Математические вопросы динамики вязкой несжимаемой жидкости, Москва, Наука, 1970.

[6] HEYWOOD J.: On uniqueness questions in the theory of viscous flow, Acta Math. 136(1976), 61-102.

[7] ЛАДЫЖЕНСКАЯ О.А., СОЛОННИКОВ В.А.: О разрешимости краевых и начально-краевых задач для уравнений Навье-Стокса в областях с некомпактными границами , Вестник ЛГУ, 13(1977), 39-47.

[8] СОЛОННИКОВ В.А., ПИЛЕЦКАС К.И.: О некоторых пространствах соленоидальных векторов и о разрешимости краевой задачи для системы уравнений Навье-Стокса в

областях с некомпактными границами, Записки научных семинаров ЛОМИ, 73(1977), 136-151.

[9] SOLONNIKOV V.A.: On the solvability of boundary and initial-boundary value problems for the Navier-Stokes system in domains with noncompact boundaries, Pacific Journal of Mathematics 81(1981), 443-458.

[10] LADYZHENSKAYA O.A., SOLONNIKOV V.A., TRUE H.: Résolution des équations de Stokes et Navier-Stokes dans les tuyaux infinis, C.R. Acad. Sci. Paris, 292(1981), Série I, 251-154.

[11] ЛАДЫЖЕНСКАЯ О.А., СОЛОННИКОВ В.А.: О начально-краевых задачах для линеаризованной системы Навье-Стокса в областях с некомпактными границами, Труды МИАН СССР 159(1982).

[12] БОГОВСКИЙ М.Е.: К L_p-теории для системы Навье-Стокса для неограниченных областей с некомпактными границами, Доклады АН СССР 255(1980), 1296-1300.

[13] СОЛОННИКОВ В.А.: Оценки решений нестационарной линеаризованной системы уравнений Навье-Стокса, Труды МИАН СССР 70(1964),213-317.

[14] СОЛОННИКОВ В.А.: Оценки решений нестационарной системы Навье-Стокса, Записки научных семинаров ЛОМИ 38(1973), 153-231.

[15] ОЛЕЙНИК О.А., РАДКЕВИЧ Е.А.: Метод введения параметра для исследования эволюционных уравнений, Успехи математических наук 33(1978), 7-76.

[16] МАКСИМОВА М.И.: Существование слабого решения параболической начально-краевой задачи в неограниченной области в классе растущих функций, Записки научных семинаров ЛОМИ 127(1983), 152-157.

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