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APPENDIX TO THE PAPER „AN EXISTENCE THEOREM FOR THE URYSOHN INTEGRAL EQUATION IN BANACH SPACES"

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Abstract: The paper contains a result concerning the Kuratowski measure of noncompactness in the space $L^1(D,E)$ of Bochner integrable functions with values in a Banach space E.

Key words: Urysohn integral equations, measures of noncompactness.

Classification: 45N05

Assume that E is a Banach space and D is a compact subset of the Euclidean space R^m. Denote by α and α_1 the Kuratowski measures of noncompactness in E and $L^1(D,E)$, respectively. Let V be a countable set of strongly measurable functions from D into E such that there exists $\mu \in L^1(D,R)$ such that $\|x(t)\| \le \mu(t)$ for all $x \in V$ and $t \in D$. For any $t \in D$ put $V(t) = \{x(t): x \in V\}$ and $v(t) = \alpha(V(t))$.

Recently Heinz [2] proved that the function v is integrable on D and

(1) $$\alpha(\{\int_T x(t)dt : x \in V\}) \le 2 \int_T v(t)dt$$

for each measurable subset T of D.

Now we shall prove the following

Theorem 1. Assume in addition that

$$\lim_{h \to 0} \sup_{x \in V} \int_D \|x(t+h) - x(t)\| dt = 0.$$

Then

$$\alpha_1(V) \le 2 \int_D v(t)dt.$$

Proof. For any positive number r put $V_r = \{x_r : x \in V\}$, where

$$x_r(t) = \frac{1}{\text{mes } Q_r} \int_{t+Q_r} x(s)ds \qquad (t \in D)$$

and Q_r is the closed ball in R^m with center 0 and radius r. It is well known that under our assumptions the set V_r is equicontinuous and uniformly bounded, and $\lim_{r\to 0} \|x - x_r\|_1 = 0$ uniformly in $x \in V$. Hence

(2) $$\alpha_1(V) = \lim_{r\to 0} \alpha_1(V_r)$$

and, by Lemma 3 of [3],

(3) $$\alpha_1(V_r) \leq \int_D \alpha(V_r(t))dt.$$

Moreover, by (1), we have

$$\alpha(V_r(t)) = \alpha(\{\frac{1}{\text{mes } Q_r} \int_{t+Q_r} x(s)ds : x \in V\}) \leq \frac{2}{\text{mes } Q_r} \int_{t+Q_r} v(s)ds,$$

so that

(4) $$\alpha(V_r(t)) \leq 2v_r(t) \quad \text{for } t \in D,$$

where $v_r(t) = \frac{1}{\text{mes } Q_r} \int_{t+Q_r} v(s)ds$. Since $\lim_{r\to 0} \|v - v_r\|_1 = 0$, from (2) - (4) it follows that $\alpha_1(V) \leq 2 \int_D v(s)ds$.

Using (1) and Theorem 1, and repeating the argument from [4], we conclude that the main result (Theorem 2) of [4] remains valid also for arbitrary Banach space E if we replace β by α and the assumption $|\lambda| < \varrho$ by $|\lambda| < \frac{1}{2}\varrho$.

References

[1] J. DANEŠ: On densifying and related mappings and their applications in nonlinear functional analysis, Theory of nonlinear operators, Akademie-Verlag, Berlin 1974, 15-56.

[2] H.P. HEINZ: On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Analysis 7 (1983), 1351-1371.

[3] S. SZUFLA: On the existence of L^p-solutions of Volterra integral equations in Banach spaces, Funkcialaj Ekvac. 27(1984).

[4] S. SZUFLA: An existence theorem for the Urysohn integral equation in Banach spaces, Comment. Math. Univ. Carolinae 25(1984), 19-27.

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