COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
25,4 (1984)

SERIES-PARALLEL GRAPHS AND WELL- AND BETTER-QUASI-ORDERINGS

Robin THOMAS

Abstract: We discuss some results concerning well- and better-quasi-ordering series-parallel graphs.

Key words and phrases: Series-parallel graph, well-quasi-ordering, better-quasi-ordering.

Classification: 05C99

The well-quasi-ordering theory (abbr. wqo) deals with sets on which a quasi-ordering (i.e. reflexive and transitive relation) is defined. Such a set Q is said to be well-quasi-ordered by a quasi-ordering $\leq$ if for any $f: \omega \longrightarrow Q$ there are $i < j$ such that $f(i) \leq f(j)$. An important quasi-ordering is "the minor" defined on the class of all graphs as follows: $G \preccurlyeq H$ if H contains a subgraph contractable onto G. Now we are able to state the so-called Wagner's conjecture, which plays a prominent role in the wqo theory.

(Conjecture) The class of all finite graphs is wqo by $\preccurlyeq$. This conjecture, if true, implies the Kuratowski's theorem for higher surfaces. x) But there are other properties of graphs, which should be useful to characterize in terms of a Kuratowski-

----

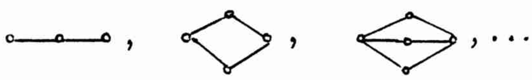
x) The proof of the Kuratowski's theorem for higher surfaces has been recently announced by Robertson and Seymour.

like theorem, perhaps for relations different from $\preccurlyeq$. In the light of this, the following theorems may be thought of as negative results.

<u>Theorem 1.</u> (i) The class of outerplanar graphs is wqo by $\preccurlyeq_c$.

(ii) The class of series-parallel graphs (= graphs which contain no subdivision of K_4) is not wqo by $\preccurlyeq_c$, where $G \preccurlyeq_c H$ if H itself can be contracted onto G.

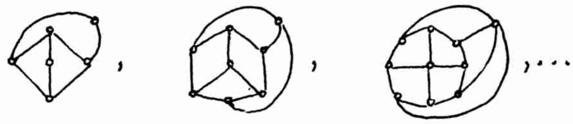
<u>Proof of (ii)</u>: The bad sequence is given by


□

<u>Theorem 2.</u> (i) The class of series-parallel graphs is wqo by $\preccurlyeq_i$

(ii) The class of planar graphs is not wqo by $\preccurlyeq_i$, where $G \preccurlyeq_i H$ if H contains an induced subgraph contractable onto G.

<u>Proof of (ii)</u>: The bad sequence is given by


□

The methods in wqo theory are based on the following well-known

<u>Key lemma</u>: If Q is wqo, then $Q^{<\omega} = \{$the set of all finite sequences of elements of Q$\}$ is wqo by the following canonical quasi-ordering (which is denoted $\leq$ as well):
$(a_1,\dots,a_n) \leq (b_1,\dots,b_m)$ if there is a strictly increasing map $f: \{1,\dots,n\} \to \{1,\dots,m\}$ such that $a_i \leq b_{f(i)}$.

<u>Proof</u>: Since now on, X, Y will always denote infinite subsets of ω . We call a sequence $f: X \to Q^{<\omega}$ good, if there

are $i < j \in X$ such that $f(i) \leq f(j)$ and we call it bad otherwise. Let $f: X \longrightarrow Q^{<\omega}$, $g: Y \longrightarrow Q^{<\omega}$. We define $f <\!* g$ if

(1) $X \subseteq Y$

(2) $f(i) \leq g(i)$ for any $i \in X$

(3) The sequence $f(i)$ is shorter than $g(i)$ for any $i \in X$.

We claim that there is a minimal (with respect to $<\!*$) bad f: $: \omega \longrightarrow Q^{<\omega}$. Indeed, choose $f(1)$ so that it is a first term of a bad sequence of elements of $Q^{<\omega}$ and the sequence $f(1)$ is the shortest possible. Then choose $f(2)$ so that $f(1)$, $f(2)$ (in that order) are first two terms of a bad sequence of elements of $Q^{<\omega}$ and the sequence $f(2)$ is the shortest possible. Continuing this process we get a bad $f: \omega \longrightarrow Q^{<\omega}$. We claim that this is the desired one. For if there is a bad $g <\!* f$, $g: X \longrightarrow$ $\longrightarrow Q^{<\omega}$, then the sequence $h: Y \longrightarrow Q^{<\omega}$ defined by

$$Y = X \cup \{i:\ i < \min X\}$$

$$h(i) = \begin{cases} f(i) & i < \min X \\ g(i) & i \in X \end{cases}$$

is bad which contradicts the choice of f.

Define

$$f_1(i) = \text{the first term of } f(i)$$

$$f_2(i) = \text{the rest of } f(i).$$

Clearly

(4) $f_1 <\!* f$, $f_2 <\!* f$.

By Ramsey theorem there is an $X \subseteq \omega$ such that either $f_1(i) \leq f_1(j)$ for any $i < j \in X$ or $f_1(i) \nleq f_1(j)$ for any $i < j \in X$. The latter case is impossible since $f_1 \upharpoonright X <\!* f$ and f is minimal bad. By the same argument there is a $Y \subseteq X$ such that $f_2(i) \leq$ $\leq f_2(j)$ for any $i < j \in Y$. Fix such i, j. We have

(5) $f_1(i) \leq f_1(j)$ and $f_2(i) \leq f_2(j)$ implies $f(i) \leq f(j)$

which contradicts the badness of f. □

Sketch of the proof of Theorems 1(i) and 2(i): We are trying to imitate the proof of the Key lemma. Thus we consider mappings $f:X \longrightarrow \mathcal{G}$, $g:Y \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is the corresponding class of graphs. Then condition (3) can be replaced by

(3') f(i) has less vertices than g(i).

Sequences f_1, f_2 satisfying (4),(5) can be defined due to a characterization of series-parallel graphs - see [1]. □

The detailed proofs will appear elsewhere, for Theorem 2 see [5]. We have considered finite graphs so far, only very little is known in case of infinite graphs. Nash-Williams, inventing a new stronger concept called better-quasi-ordering (bqo) has proved that the class of trees (finite or infinite) is wqo (in fact bqo). A nice explanation of the bqo theory can be found in [4]. Using this theory and ideas of Laver [2] we obtained

Theorem 3. The class of all (finite or infinite) series-parallel graphs is wqo (in fact bqo) by $\preccurlyeq$.

The proof of Theorem 3 is based on a characterization of (infinite) series-parallel graphs, which is in the spirit of Laver's scattered type characterization [2]. We are not going to state this theorem here, because it requires some additional definitions. Another important feature of the proof of Theorem 3 is that any series-parallel graph can be written as a countable union of series-parallel graphs, each of them contains no infinite path. That is an easy consequence of our characterization theorem for series-parallel graphs. The details will appear elsewhere.

References

[1] R.J. DUFFIN: Topology of series-parallel networks, J. Math. Anal. and Appl. 10(1965), 303-318.

[2] R. LAVER: Better-quasi-orderings and a class of trees, Studies in foundations and combinatorics, Adv. in math. supplementary studies 1(1978), 31-48.

[3] C.St.J.A. NASH-WILLIAMS: On well-quasi-ordering infinite trees, Proc. Cambridge Phil. Soc. 61(1965), 697-720.

[4] S.G. SIMPSON: BQO theory and Fraisse´s conjecture, appendix to Descriptive Set Theory by R.B. Mansfield and G. Weitkamp, Oxford Logic Series

[5] R. THOMAS: Graphs without K_4 and well-quasi-ordering, to appear in J. Comb. Theory (B).

Matematicko-fyzikální fakulta, Univerzita Karlova, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(Oblatum 10.6. 1984)