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TORSION QUASIMODULES

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Abstract: Using the preradical approach, torsion and cocyclic quasimodules are investigated. It is also shown how varieties of quasimodules are constructed from varieties of modules and 3-elementary commutative Moufang loops.

Key words: Commutative Moufang loop, quasimodule, preradical, variety of quasimodules.

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1. Introduction

A loop $Q(+)$ satisfying the identity $(x+x)+(y+z) = (x+y)+(x+z)$ is commutative and it is called a commutative Moufang loop. We denote by $\underline{C}(Q(+))$ the centre of $Q(+)$, i.e. $a \in \underline{C}(Q(+))$ iff $(a+x)+y = a+(x+y)$ for all $x,y \in Q$. Then $\underline{C}(Q(+))$ is a normal subloop of $Q(+)$, $3x \in \underline{C}(Q(+))$ for every $x \in Q$ and we have the upper central series $0 = \underline{C}_0(Q(+)) \subseteq \underline{C}_1(Q(+)) \subseteq \underline{C}_2(Q(+)) \subseteq \ldots \subseteq \underline{C}_n(Q(+)) \subseteq \ldots$ of $Q(+)$, where $\underline{C}_{n+1}(Q(+))/\underline{C}_n(Q(+)) = \underline{C}(Q(+)/\underline{C}_n(Q(+)))$ for every $n = 0,1,2,\ldots$. The loop $Q(+)$ is said to be nilpotent of class at most n if $\underline{C}_n(Q(+)) = Q$. Further, for all $x,y,z \in Q$, the associator $[x,y,z]$ is defined by $[x,y,z] = ((x+y)+z) - (x+(y+z))$ and we denote by $\underline{A}(Q(+))$ the subloop generated by all associators. Then $\underline{A}(Q(+))$ is a normal subloop of $Q(+)$ and $3x = 0$ for every $x \in \underline{A}(Q(+))$. Moreover, we have the lower central series $Q = \underline{A}_0(Q(+)) \supseteq \underline{A}_1(Q(+)) \supseteq \underline{A}_2(Q(+)) \supseteq \ldots \supseteq \underline{A}_n(Q(+)) \supseteq \ldots$ of $Q(+)$, where $\underline{A}_{n+1}(Q(+))$ is the subloop generated by all associators $[x,y,z]$, $x \in \underline{A}_n(Q(+))$, $y,z \in Q$, for every $n = 0,1,2,\ldots$ The loop $Q(+)$ is nilpotent of class at most n iff $\underline{A}(Q(+)) \subseteq \underline{C}_{n-1}(Q(+))$ iff $\underline{A}_{n-1}(Q(+)) \subseteq \underline{C}(Q(+))$ and iff $\underline{A}_n(Q(+)) = 0$. As for details and further information concerning commutative Moufang loops, the reader is referred to [5].

Let $Q(+)$ be a commutative Moufang loop. A mapping f of Q into Q is said to be n-central, n being an integer, if $nx + f(x) \in \underline{C}(Q(+))$ for every $x \in Q$. Clearly, f is n-central iff it is m-central, where $m \in \{0,1,2\}$ and $n = 3k+m$. The zero endomorphism $x \to 0$ is 0-central, the automorphism $x \to -x$ is 1-central and the identical automorphism $x \to x$ is 2-central. As proved in [9], the set of all (0,1,2-)central endomorphisms of $Q(+)$ is an associative ring with unit.

Throughout the paper, let $\underline{R}$ be an associative ring with unit, Φ a ring homomorphism of $\underline{R}$ onto the three-element field $\underline{Z}_3 = \{0,1,2\}$ and $\underline{I} = \mathrm{Ker}\ \Phi$. By a ($\Phi$-special unitary left $\underline{R}$-) quasimodule Q we mean a commutative Moufang loop $Q(+)$ equipped with scalar multiplication by elements of $\underline{R}$ such that the usual module identities are satisfied, i.e. $r(x+y) = rx+ry$, $(r+s)x = rx+sx$, $r(sx) = (rs)x$, $1x = x$ for all $r,s \in \underline{R}$, $x,y \in Q$ and, moreover, $tx \in \underline{C}(Q(+))$ for all $x \in Q$ and $t \in \underline{I}$. The last condition says that the endomorphism $x \to rx$ of $Q(+)$

is $(-\Phi(r))$-central for all $r \in \underline{R}$. Some information concerning quasimodules and constructions of non-associative quasimodules can be found in [9], [10] and [11].

Let Q be a quasimodule. A subquasimodule P of Q is normal in Q (i.e. P is a block of a congruence of Q) iff $P(+)$ is a normal subloop of $Q(+)$. Now it is easy to see that all the members of the upper central series as well as of the lower central series of $Q(+)$ are normal subquasimodules of Q. Hence Q is said to be nilpotent of class at most n iff the loop $Q(+)$ is so. Further, we shall say that Q is a primitive quasimodule if $rx = 0$ for all $r \in \underline{I}$ and $x \in Q$.

1.1 <u>Example</u>. Every commutative Moufang loop (abelian groups included) is a $\underline{Z}$-quasimodule, $\underline{Z}$ being the ring of integers and Φ the natural homomorphism of $\underline{Z}$ onto $\underline{Z}_3$.

1.2 <u>Example</u>. Let $Q(+)$ be a 3-elementary commutative Moufang loop. Put $rx = \Phi(r)x$ for all $r \in \underline{R}$ and $x \in Q$. Then $Q = Q(+,rx)$ is a primitive quasimodule and we see that the classes of primitive quasimodules, $\underline{Z}_3$-quasimodules and 3-elementary commutative Moufang loops are equivalent.

1.3 <u>Example</u>. Let $Q(+)$ be a non-associative commutative Moufang loop. Denote by R the ring of central endomorphisms of $Q(+)$. For every $f \in R$ there is a unique $n(f) \in \{0,1,2\}$ such that f is $n(f)$-central and the mapping $f \to -n(f)$ is a ring homomorphism of R onto $\underline{Z}_3$. Now, Q has an R-quasimodule structure.

1.4 <u>Example</u>. A quasigroup G is said to be trimedial if every subquasigroup of G generated by at most 3 elements is medial, i.e. satisfies the identity $xy.uv = xu.yv$. Trimedial and medial quasigroups appear in many geometrical situations (see e.g. [1], [4], [15], [16]) and important classes of trimedial quasigroups are idempotent trimedial quasigroups (called also distributive quasigroups and determined by the identities $x.yz = xy.xz$, $yz.x = yx.zx$), symmetric trimedial quasigroups (better known as CH-quasigroups or Manin quasigroups and determined by the identities $xy = yx$, $x.xy = y$ and $xx.yz = xy.xz$) and idempotent symmetric trimedial quasigroups (distributive Steiner quasigroups known in an equivalent form as Hall triple systems). Now, let $R = \underline{Z}[x,y,x^{-1},y^{-1}]$. As proved in [12], the classes of pointed trimedial quasigroups and centrally pointed quasimodules are equivalent.

1.5 Proposition. Let n be a positive integer.
(i) Every quasimodule which can be generated by at most n elements is nilpotent of class at most $m = \max(1,n-1)$.
(ii) The free primitive quasimodule of rank n (and hence the free quasimodule of rank n) is nilpotent of class precisely m.

Proof. (i) See [9, Proposition 4.3]; the assertion is a consequence of the same result for commutative Moufang loops which is known as the Bruck-Slaby's theorem ([5, Theorem VIII,10.1]).
(ii) See [2, Corollary IV.3.2].

1.6 Proposition. Let Q be a quasimodule. Then both $\underline{A}(Q)$ and $Q/\underline{C}(Q)$ are primitive.

Proof. Let $r \in \underline{I}$. We have $rx \in \underline{C}(Q)$ for all $x \in Q$ and it is clear that $Q/\underline{C}(Q)$ is primitive. On the other hand, the mapping

$f: x \to rx$ is an endomorphism of $Q(+)$ and $\mathrm{Im}\, f \subseteq \underline{C}(Q(+))$. Consequently, $\mathrm{Im}\, f$ is associative, hence $\underline{A}(Q) \subseteq \mathrm{Ker}\, f$ and $ry = 0$ for all $y \in \underline{A}(Q)$.

1.7 __Proposition__. (i) Every simple quasimodule is a module.
(ii) Every maximal subquasimodule of a nilpotent quasimodule is normal.
(iii) If the ring $\underline{R}$ is left noetherian then every subquasimodule of a finitely generated quasimodule is finitely generated.

Proof. See [9, Lemma 4.8, Corollary 4.11, Proposition 4.6] .

Let Q be a quasimodule. For all $a,b \in Q$, define a mapping $i_{a,b}$ by $i_{a,b}(x) = ((x+a)+b) - (a+b)$. Then $i_{a,b}$ is an automorphism of the loop $Q(+)$ and $i_{a,b}(x) = x + [x,a,b]$.

1.8 __Lemma__. Let P be a subquasimodule of a quasimodule Q . The following conditions are equivalent:

(i) P is a normal subquasimodule of Q .
(ii) $i_{a,b}(P) \subseteq P$ for all $a,b \in Q$.
(iii) $[x,a,b] \in P$ for all $x \in P$, $a,b \in Q$.

Proof. Easy.

1.9 __Lemma__. Let Q be a quasimodule and $a,b \in Q$. Then $i_{a,b}$ is an automorphism of the quasimodule Q .

Proof. Let $r \in \underline{R}$ be arbitrary and $s = -\Phi(r).1$. We have $(r+s)x \in \underline{C}(Q)$ for every $x \in Q$. Denote $c = (r+s)a$, $d = (r+s)b$. Then $si_{a,b}(x) + i_{a,b}(rx) = i_{a,b}((r+s)x) = (r+s)x$ and $(r+s)i_{a,b}(x) = i_{c,d}((r+s)x) = (r+s)x$. Consequently, $i_{a,b}(rx) = ri_{a,b}(x)$.

2. Preradicals

By a preradical p (for quasimodules) we mean any subfunctor of the identity functor, i.e. p assigns to each quasimodule Q a subquasimodule $p(Q)$ in such a way that $f(p(Q)) \subseteq \subseteq p(P)$ whenever f is a homomorphism of Q into a quasimodule P. The basic properties of preradicals for quasimodules are the same as in the module case and the reader is referred to [3] and [9] for details. We shall also use the terminology introduced in [3]. Recall that a preradical p is said to be hereditary if $p(P) = P \cap p(Q)$ whenever P is a subquasimodule of a quasimodule Q. A preradical p is said to be cohereditary if $f(p(Q)) = p(P)$ whenever f is a surjective homomorphism of a quasimodule Q onto a quasimodule P. If p is a preradical then by 1.9 $p(Q)$ is a normal subquasimodule of Q for every quasimodule Q. Further, p is said to be a radical if $p(Q/p(Q)) = 0$ for every quasimodule Q.

2.1 <u>Example</u>. It is easy to see that for every integer $n \geq 0$, $\underline{A}_n$ is a cohereditary radical. On the other hand, $\underline{C}$ is not a preradical, since the centre is in general preserved only by surjective homomorphisms.

2.2 <u>Example</u>. For every quasimodule Q, let $\underline{B}(Q)$ denote the least normal subquasimodule of Q such that the corresponding factor is primitive. Then $\underline{B}$ is a cohereditary radical. By 1.6, $\underline{B}(Q) \subseteq \underline{C}(Q)$.

2.3 <u>Lemma</u>. Let Q be a quasimodule generated by a set M. Then

$\underline{B}(Q)$ is just the subloop of $Q(+)$ generated by all rx , $r \in \underline{I}$, $x \in M$.

Proof. Denote by P that subloop. Since $\underline{I}$ is an ideal, it is easy to verify that P is a subquasimodule and $ry \in P$ for all $r \in \underline{I}$, $y \in Q$. Further, P is normal and hence $P = \underline{B}(Q)$.

2.4 Example. For every quasimodule Q , let $\underline{D}(Q)$ denote the least normal subquasimodule such that the corresponding factor is a $\underline{Z}_3$-module, i.e. a vector space over $\underline{Z}_3$. Then $\underline{D}$ is a co-hereditary radical and, moreover, $\underline{D} = \underline{A} + \underline{B}$, i.e. $\underline{D}(Q) = \{ x+y \ ; \ x \in \underline{A}(Q) \ , \ y \in \underline{B}(Q) \}$ for every quasimodule Q .

2.5 Example. For every quasimodule Q , let $\underline{J}(Q)$ denote the intersection of all maximal normal subquasimodules of Q ; $\underline{J}(Q) = Q$ if there are no such subquasimodules. Clearly, $\underline{J}(Q)$ is just the intersection of all $\mathrm{Ker}\, f$, f ranging over all homomorphisms of Q into simple (quasi)modules. Thus $\underline{J}$ is a radical and $\underline{A} \subseteq \underline{J} \subseteq \underline{D}$ (use 1.7).

2.6 Proposition. Let Q be a quasimodule.

(i) $\underline{J}(Q)$ is the intersection of all normal maximal subquasimodules of Q .

(ii) If Q is nilpotent then $\underline{J}(Q)$ is the intersection of all maximal subquasimodules of Q .

(iii) Let Q be finitely generated, $P \subseteq \underline{J}(Q)$ be a normal subquasimodule of Q and let f denote the natural homomorphism of Q onto Q/P . If M is a subset of Q such that $f(M)$ generates Q/P then Q is generated by M .

Proof. (i) and (ii) follow from 1.7(i),(ii), respectively.

(iii) Assume, on the contrary, that Q is not generated by M and let N be a finite set generating Q. Further, let K be a subset of N maximal with respect to the property that $M \cup K$ do not generate Q and take $a \in N \setminus K$. There is a subquasimodule G of Q maximal with respect to $M \cup K \subseteq G$ and $a \notin G$. It is easy to see that G is a maximal subquasimodule of Q and hence $P \subseteq G$, a contradiction.

Let $\mathcal{F}$ be a filter of left ideals of the ring $\underline{R}$. For every quasimodule Q, let $p(Q)$ denote the set of all $x \in Q$ such that $(0:x) = \{r \in \underline{R} ; rx = 0\} \in \mathcal{F}$. Then $p = p_{\mathcal{F}}$ is a hereditary preradical.

2.7 <u>Proposition</u>. There is a one-to-one correspondence between hereditary preradicals and filters of left ideals given by

$$\mathcal{F} \longrightarrow p\,(Q) = \{x \in Q \,;\, (0:x) \in \mathcal{F}\}\ ,$$

$$p \longrightarrow \mathcal{F}_p = \{I \subseteq \underline{R} \,;\, p(\underline{R}/\underline{I}) = \underline{R}/\underline{I}\}\ .$$

This correspondence induces a one-to-one correspondence between hereditary radicals and radical filters.

Proof. See [9, Proposition 3.2, Lemma 3.3, Lemma 3.4].

Let p be a preradical. Define a preradical $\hat{p}$ by $\hat{p}(Q) = \bigcap \operatorname{Ker} f$, $f: Q \to P$, $p(P) = 0$. Clearly, $\hat{p}$ is a radical and it is just the least radical containing p.

2.8 <u>Lemma</u>. Let p be a preradical. Then a quasimodule Q is $\hat{p}$-torsion iff there are an ordinal number α and a chain Q_β, $0 \leq \beta \leq \alpha$, of normal subquasimodules of Q such that $Q_0 = 0$, $Q_\alpha = Q$ and $Q_{\beta+1}/Q = p(Q/Q_\beta)$ for every $0 \leq \beta < \alpha$, $Q_\beta = \bigcup_{\gamma<\beta} Q_\gamma$ for β limit.

Proof. Obvious.

2.9 Lemma. Let p be a hereditary preradical. Then $\hat{p}$ is a hereditary radical.

Proof. See [9, Proposition 3.7].

Let A be a simple module. Then A is isomorphic to $\underline{R}/I$ for a maximal left ideal I; we denote by $\mathcal{F} = \mathcal{F}_I$ ($\mathcal{R} = \mathcal{R}_I$) the filter (radical filter) generated by I and we put $\underline{S}_A = p_{\mathcal{F}}$. By 2.7 and 2.9, $\hat{\underline{S}}_A = p_{\mathcal{R}}$.

The field $\underline{Z}_3$ considered as a module is simple and isomorphic to $\underline{R}/\underline{I}$. We shall also use the notation $\underline{L} = \underline{S}_{\underline{Z}_3}$ and $\underline{K} = \hat{\underline{L}}$. Finally, denote by $\mathcal{F}$ (resp. $\mathcal{R}$) the filter (radical filter) generated by all maximal left ideals and put $\underline{S} = p_{\mathcal{F}}$, so that $\hat{\underline{S}} = p_{\mathcal{R}}$.

3. $\underline{S}$- and $\hat{\underline{S}}$-torsion quasimodules

3.1 Proposition. A quasimodule Q is $\underline{L}$-torsion iff it is primitive.

Proof. Obvious.

3.2 Proposition. Let Q be a finitely generated primitive quasimodule. Then Q is finite and $|Q| = 3^n$ for some $n \geq 0$.

Proof. The field $\underline{Z}_3$ is clearly a noetherian ring and the result follows from 1.7(iii) by induction on the nilpotence class of Q.

3.3 Proposition. For every quasimodule Q, $\underline{A}(Q) \subseteq \underline{L}(Q) \subseteq \underline{K}(Q) \subseteq \subseteq \hat{\underline{S}}(Q)$ and $\underline{A}(Q) \subseteq \underline{L}(Q) \subseteq \underline{S}(Q)$. Consequently, every $\underline{K}$-torsionfree quasimodule (and also every $\hat{\underline{S}}$-torsionfree quasimodule) is a module.

Proof. This follows from 1.6.

3.4 Corollary. Let A be a simple module not isomorphic to $\underline{Z}_3$. Then every $\hat{\underline{S}}_A$-torsion quasimodule is a module.

Now, denote by $\mathcal{Y}$ a representative set of simple modules such that $\underline{Z}_3 \in \mathcal{Y}$.

3.5 Proposition. Let Q be an $\underline{S}$-torsion quasimodule. Then Q is a direct sum of subquasimodules $\underline{S}_A(Q)$, $A \in \mathcal{Y}$. If $A \neq \underline{Z}_3$ then $\underline{S}_A(Q)$ is a module isomorphic to a direct sum of copies of A. If $A = \underline{Z}_3$ then $\underline{S}_A(Q)$ is a primitive quasimodule.

Proof. First, let $B \in \mathcal{Y}$ be arbitrary and let P be the subquasimodule generated by $\bigcup \underline{S}_A(Q)$, $A \in \mathcal{Y}$, $A \neq B$. Let $\mathcal{F}$ be the filter generated by all maximal left ideals I such that $\underline{R}/I$ is not isomorphic to B and let $a \in \underline{S}_B(Q) \cap P$. Then the cyclic module $\underline{R}a$ is both $\underline{S}_B$-torsion and $p_{\mathcal{F}}$-torsion (both $\underline{S}_B$ and $p_{\mathcal{F}}$ are hereditary and P is $p_{\mathcal{F}}$-torsion), so that $a = 0$. Now, suppose that $B = \underline{Z}_3$. Then $(P+\underline{C}(Q))/\underline{C}(Q)$ is both $\underline{L}$-torsion and $p_{\mathcal{F}}$-torsion, hence it is a zero module and $P \subseteq \underline{C}(Q)$. In particular, P is a module and the sum $\underline{L}(Q)+P$ is direct. Finally, $\underline{A}(Q) \subseteq \underline{L}(Q)$ and $Q/\underline{A}(Q) = (\underline{L}(Q)+P)/\underline{A}(Q)$. From this, $Q = \underline{L}(Q)+P$ and the rest is clear.

3.6 Theorem. Suppose that the ring $\underline{R}$ has primary decompositions. Let Q be an $\hat{\underline{S}}$-torsion quasimodule. Then Q is a direct sum of subquasimodules $\hat{\underline{S}}_A(Q)$, $A \in \mathcal{Y}$. If $A \neq \underline{Z}_3$ then $\hat{\underline{S}}_A(Q)$ is a module.

Proof. We have $\underline{A}(Q) \subseteq \underline{L}(Q)$ and $Q/\underline{A}(Q)$ is generated by the image of $\bigcup \hat{\underline{S}}_A(Q)$, $A \in \mathcal{Y}$. Hence Q is generated by this set and we can proceed in the same way as in the proof of 3.5.

3.7 Proposition. Let Q be a finite $\underline{K}$-torsion module. Then $|Q| =$

$= 3^n$ for some $n \geq 0$.

Proof. The assertion is an easy consequence of 3.2.

3.8 Lemma. Let I be an ideal of $\underline{R}$ and let $\mathcal{R}$ be the radical filter generated by I . Then:

(i) A left ideal K belongs to $\mathcal{R}$ iff for every sequence $a_1, a_2, \ldots$ of elements of I there is $n \geq 1$ with $a_n \ldots a_1 \in K$.

(ii) If I is finitely generated as a left ideal then a left ideal K belongs to $\mathcal{R}$ iff $I^n \subseteq K$ for some $n \geq 1$.

Proof. See e.g. [3, Corollary III.4.6, Proposition III.4.4] .

3.9 Corollary. Let Q be a quasimodule. Then $x \in \underline{K}(Q)$ iff for every sequence $a_1, a_2, \ldots$ of elements of $\underline{I}$ there is $n \geq 1$ with $a_n \ldots a_1 x = 0$. Moreover, if $\underline{I}$ is finitely generated as a left ideal then $x \in \underline{K}(Q)$ iff $\underline{I}^n x = 0$ for some $n \geq 1$.

3.10 Lemma. Let I be a finitely generated maximal left ideal of the ring $\underline{R}$ such that I is an ideal and $A = \underline{R}/I$ is finite. Then every finitely generated $\hat{\underline{S}}_A$-torsion module is finite.

Proof. Clearly, I^n/I^{n+1} is finitely generated and $\underline{R}/I^n$ is finite for every $n \geq 1$. By 3.8(ii), every cyclic $\hat{\underline{S}}_A$-torsion module is finite and the rest is clear.

3.11 Propositinn. Suppose that $\underline{I}$ is finitely generated as a left ideal. Then every finitely generated $\underline{K}$-torsion quasimodule Q is finite.

Proof. We shall proceed by induction on the nilpotence class n of Q . If $n \leq 1$ then Q is a module and the result follows from 3.10. Now, let $n \geq 2$. We have $\underline{A}_n(Q) = 0$, $\underline{A}_{n-1}(Q) \subseteq \underline{C}(Q)$ and $G = Q/\underline{A}_{n-1}(Q)$ is finite by the induction hypothesis. There are two finite subsets N and M of $\underline{A}_{n-2}(Q)$ and Q , respectively, such that $(N+\underline{A}_{n-1}(Q))/\underline{A}_{n-1}(Q) = \underline{A}_{n-2}(Q)/\underline{A}_{n-1}(Q)$ and $(M+\underline{A}_{n-1}(Q))/\underline{A}_{n-1}(Q) = G$. Denote by P the subquasimodule generated by all the associators $[x,y,z]$, $x \in N$, $y,z \in M$. Then P is a finitely generated subquasimodule of $\underline{A}_{n-1}(Q)$ and hence P is a normal finitely generated submodule of Q. In particular, P is finite. On the other hand, if $u \in \underline{A}_{n-2}(Q)$ and $v,w \in Q$ are arbitrary, then $u = x+a$, $v = y+b$, $w = z+c$ for some $x \in N$, $y,z \in M$ and $a,b,c \in \underline{C}(Q)$. We have $[u,v,w] = [x,y,z] \in P$ and we see that $P = \underline{A}_{n-1}(Q)$. Thus both $\underline{A}_{n-1}(Q)$ and G are finite, so that Q is finite, too.

3.12 Proposition. Let I be a finitely generated maximal left ideal of $\underline{R}$ such that I is an ideal and $A = \underline{R}/I$ is finite. Then every finitely generated $\underline{\hat{S}}_A$-torsion quasimodule is finite.

Proof. By 3.4, 3.10 and 3.11.

3.13 Theorem. Suppose that every maximal left ideal of $\underline{R}$ is an ideal, finitely generated as a left ideal, maximal ideals commute and every simple module is finite. Let Q be a finitely generated $\underline{\hat{S}}$-torsion quasimodule. Then Q is finite and there are $A_1,\dots,A_n \in \mathcal{Y}$ such that Q is isomorphic to the product $\underline{\hat{S}}_{A_1}(Q) \times \dots \times \underline{\hat{S}}_{A_n}(Q)$.

Proof. The ring $\underline{R}$ has primary decompositions and the result now follows from 3.6 and 3.12.

3.14 Remark. The assumptions of the preceding theorem are satisfied e.g. if $\underline{R}$ is a finitely generated commutative ring.

3.15 Proposition. Suppose that $\underline{R}$ is left noetherian and every simple module is finite. Then every finitely generated $\underline{\hat{S}}$-torsion quasimodule is finite.

Proof. In the situation of Lemma 2.8, α is finite by 1.7(iii) and we can proceed by induction, using 3.5 and 3.2.

4. Cocyclic quasimodules

A quasimodule Q is said to be cocyclic if it contains a (non-zero) normal simple submodule A such that A is contained in every non-zero normal subquasimodule of Q.

4.1 __Lemma__. Let Q be a quasimodule and A be a normal simple subquasimodule of Q. Then $A \subseteq \underline{C}(Q)$.

Proof. Let $a \in A$ and $x,y \in Q$ be arbitrary. Denote by P the subquasimodule generated by a,x,y. Then P is a nilpotent quasimodule and $A \subseteq \underline{C}(P)$ by [9, Lemma 4.7]. Consequently, $(a+x)+y = a+(x+y)$ and we have proved that $A \subseteq \underline{C}(Q)$.

4.2 __Proposition__. Let Q be a cocyclic quasimodule and A be the normal simple submodule of Q. Then:

(i) $A \subseteq \underline{C}(Q)$ and $\underline{\hat{S}}(Q) = \underline{\hat{S}}_A(Q)$.

(ii) Q is subdirectly irreducible.

(iii) Either $A \subseteq \underline{A}(Q)$ and A is isomorphic to $\underline{Z}_3$ or Q is a module.

(iv) $\underline{C}(Q)$ is a cocyclic module.

Proof. Easy (use 4.1).

4.3 __Corollary__. A quasimodule Q is cocyclic iff $\underline{C}(Q) \neq 0$ and Q is subdirectly irreducible. In particular, a nilpotent (resp. finitely generated) quasimodule is cocyclic iff it is subdirectly irreducible.

4.4 __Proposition__. Suppose that $\underline{R}$ is commutative and noetherian. Let Q be a cocyclic quasimodule and A the normal simple submodule of Q. Then:

(i) Q is $\underline{\mathbb{S}}_A$-torsion.

(ii) If Q is finitely generated and A is finite then Q is finite.

(iii) If Q is non-associative then A is isomorphic to $\underline{Z}_3$ and Q is $\underline{\mathbb{S}}$-torsion.

(iv) If Q is finitely generated and non-associative then Q is finite.

Proof. First, let Q be a module. By [3, Proposition VI.3.4], $\underline{R}$ is a stable ring and so the injective hull E of Q is $\underline{\mathbb{S}}_A$-torsion. Now, suppose that A is isomorphic to $\underline{Z}_3$. We have $A \subseteq \underline{C}(Q)$ and $\underline{C}(Q)$ is $\underline{\mathbb{S}}$-torsion, since it is a cocyclic module. On the other hand, $Q/\underline{C}(Q)$ is a primitive quasimodule and thus Q is $\underline{\mathbb{S}}$-torsion. The rest is clear.

4.5 __Example__. Let α be an infinite limit ordinal number and $N = \{a_0, a_1, \dots\}$ be the canonical basis of the vector space $Q = \underline{Z}_3^{(\alpha)}$. Define a mapping t of N^3 into Q by $t(a_\beta, a_{\beta+1}, a_{\beta+2}) = a_0$, $t(a_{\beta+1}, a_\beta, a_{\beta+2}) = -a_0$ for $1 \leq \beta \leq \alpha$ and $t(a_\beta, a_\gamma, a_\delta) = 0$ in all remaining cases. It is clear that t can be extended uniquely to a trilinear mapping T of Q^3 into Q such that $T(x,x,y) = T(T(x,y,z),u,v) = T(u,v,T(x,y,z)) = T(u,T(x,y,z),v) = 0$ for all $x,y,z,u,v \in Q$. Put $x * y = x + y + T(x,y,x-y)$ for all $x,y \in Q$. Then $Q' = Q(*)$ is a primitive quasimodule nilpotent of class 2 (see [M]). Moreover, $a \in \underline{C}(Q')$ iff $T(a,x,y)+T(x,y,a)+T(y,a,x) = 0$ for all $x,y \in Q$. Now it is easy to check that we have $\underline{C}(Q') = \underline{A}(Q') = \{0, a_0, -a_0\}$. In particular, Q' is a cocyclic quasimodule. Thus for every infinite cardinal ε there is a cocyclic primitive quasimodule (nilpotent of class 2) of cardinality ε.

4.6 __Example__. Let $n \geq 4$, $Q = \underline{Z}_3^{(n)}$, $a_1 = (1,0,\dots,0)$, ..., $a_n =$

$= (0,\dots,0,1)$, $N = \{ a_1,\dots,a_n \}$. Define a mapping t of N^3 into Q by $t(a_i,a_{i+1},a_{i+2}) = a_n$, $t(a_{i+1},a_i,a_{i+2}) = -a_n$ for every $1 \le i \le n-3$, $t(a_{n-2},a_{n-1},1) = a_n$, $t(a_{n-1},a_{n-2},1) = -a_n$, $t(a_{n-1},1,2) = a_n$, $t(1,a_{n-1},2) = -a_n$. Then t can be extended uniquely to a trilinear mapping T of Q^3 into Q and we put $x * y = x+y+T(x,y,x-y)$. Then $Q' = Q(*)$ is a primitive quasimodule nilpotent of class 2 , $|Q'| = 3^n$ and it is not difficult to check that Q' is cocyclic, provided $n \neq 5$ and $n \neq 6k+1$. By [14], for every $m \ge 1$, $m \neq 2,3,5$, there is a cocyclic primitive quasimodule of order 3^m , nilpotent of class 2. On the other hand, it is clear that there are no cocyclic primitive quasimodules of order 3^2, 3^3 and it is proved in [8] that there is no cocyclic primitive quasimodule of order 3^5 .

5. Cohereditary radicals and varieties of quasimodules

By a variety of quasimodules we mean a non-empty class of quasimodules closed under cartesian products, subquasimodules and homomorphic images.

5.1 Proposition. (i) If q is a cohereditary radical then the class $\mathcal{V}_q$ of all torsionfree quasimodules is a variety.
(ii) Let $\mathcal{V}$ be a variety of quasimodules. For every quasimodule Q , let $q_{\mathcal{V}}(Q) = \bigcap \operatorname{Ker} f$, $f: Q \to P$, $P \in \mathcal{V}$. Then $q_{\mathcal{V}}$ is a cohereditary radical.
(iii) The correspondence $q \to \mathcal{V}_q$ and $\mathcal{V} \to q_{\mathcal{V}}$ between cohereditary radicals and varieties of quasimodules is bijective.

Proof. Easy.

Let $\mathcal{V}$ be a variety of quasimodules. Denote by $\mathcal{V}_m$ (resp. $\mathcal{V}_p$) the class of all modules (resp. primitive quasimodules) contained in $\mathcal{V}$ and put $L_{\mathcal{V}} = q_{\mathcal{V}}(R)$. Then both $\mathcal{V}_m$ and $\mathcal{V}_p$

are varieties, $L_{\mathcal{V}}$ is an ideal of $\underline{R}$, $L_{\mathcal{V}}Q = 0$ for every quasimodule $Q \in \mathcal{V}$ and a module M belongs to $\mathcal{V}_m$ iff $L_{\mathcal{V}}M = 0$.

5.2 Proposition. Let $\mathcal{V}$ be a variety of quasimodules such that $L_{\mathcal{V}} \nsubseteq \underline{I}$. Then $\mathcal{V} = \mathcal{V}_m$ and $\mathcal{V}_p = 0$.

Proof. We have $\underline{R} = L_{\mathcal{V}} + \underline{I}$, so that $Q = \underline{R}Q = 0$ for every $Q \in \mathcal{V}_p$.

5.3 Proposition. Let $\mathcal{V}$ be a variety of quasimodules and let $F \in \mathcal{V}$ be a quasimodule free in $\mathcal{V}$. Then $\underline{B}(F) \cap \underline{A}(F) = 0$.

Proof. Let X be a free basis of F and let f denote the natural homomorphism of F onto $G = F/\underline{A}(F)$. Then G is a free $\underline{R}/L_{\mathcal{V}}$-module, $f \mid X$ is injective and $f(X)$ is a free basis of G . Now, let $a \in \underline{A}(F) \cap \underline{B}(F)$. By 2.3 there are $n \geq 0$, pairwise different $x_1,\dots,x_n \in X$ and elements $r_1,\dots,r_n \in \underline{I}$ with $a = r_1x_1+\dots+r_nx_n$ (we have $r_ix_i \in \underline{C}(F)$). Consequently, $0 = r_1f(x_1)+\dots+r_nf(x_n)$, $r_1,\dots,r_n \in L$ and $a = 0$.

5.4 Proposition. Let $\mathcal{V}$ be a variety of quasimodules. Then $\mathcal{V}$ is just the variety generated by $\mathcal{V}_m \cup \mathcal{V}_p$.

Proof. This is an easy consequence of 5.3.

5.5 Proposition. Let $\mathcal{U}$ and $\mathcal{W}$ be varieties of modules and primitive quasimodules, respectively. Denote by $\mathcal{V}$ the variety of quasimodules generated by $\mathcal{U} \cup \mathcal{W}$. Then $\mathcal{V}_m = \mathcal{U}$ and $\mathcal{V}_p = \mathcal{W}$.

Proof. Let $F \in \mathcal{V}$ be a free quasimodule of infinite countable rank. Since $\mathcal{V}$ is generated by $\mathcal{U} \cup \mathcal{W}$, F is isomorphic to a subquasimodule of the product $G \times P$, $G \in \mathcal{U}$ and $P \in \mathcal{W}$ being free of infinite countable rank; we shall assume that F is a subquasimodule ˘ ^ x P . Consequently, $L_{\mathcal{U}}F = 0$ and we see that $\mathcal{U} = \mathcal{V}_m$. On the other hand, $\underline{B}(F) \subseteq H = G \times 0$, $\underline{B}(F)$ is a normal subquasimodule of $G \times P$ and $F/\underline{B}(F)$ is isomorphic to a subquasimodule of $(H/\underline{B}(F)) \times P \in \mathcal{W}$. However, $\mathcal{V}_p$ is generated by $F/\underline{B}(F)$ and therefore $\mathcal{W} = \mathcal{V}_p$.

Now, denote by $\mathcal{I}$ the dual lattice of the lattice of ideals of the ring $\underline{R}$ and by $\mathcal{P}$ the lattice of varieties of primitive quasimodules (i.e. the lattice of varieties of 3-elementary commutative Moufang loops). Let $\mathcal{L}$ be the subset of $\mathcal{I} \times \mathcal{P}$ formed by all couples $(I, \mathcal{U})$, where either $\mathcal{U} = 0$, or $0 \neq \mathcal{U} \neq \mathcal{U}_m$ and $I \subseteq \underline{I}$.

5.6 <u>Theorem</u>. The lattice of varieties of quasimodules is isomorphic to the lattice $\mathcal{L}$.

Proof. Apply 5.2, 5.4 and 5.5.

5.7 <u>Proposition</u>. Let $\underline{R}$ be left noetherian, $n \geq 0$ and $\mathcal{V}$ be a variety of quasimodules nilpotent of class at most n. Then $\mathcal{V}$ is finitely based (i.e. $\mathcal{V}$ can be determined by a finite number of identities).

Proof. Using 1.7(iii), we can proceed in the same way as in the proof of [6, Theorem III].

5.8 <u>Corollary</u>. Let $\underline{R}$ be left noetherian, $n \geq 0$ and $\mathcal{V}$ be a variety of quasimodules nilpotent of class at most n. Then $\mathcal{V}$ contains only countably many subvarieties.

By [13, § 10], the lattice of varieties of primitive quasimodules nilpotent of class at most 2 is a three-element chain. Having some information on the lattice of ideals of $\underline{R}$ (e.g. if $\underline{R}$ is a commutative principal ideal ring, etc.) and using 5.6, we can describe the lattice of varieties of quasimodules nilpotent of class at most 2 . Moreover, applying the methods developed in [7] for medial quasigroups, the results are transferable to various classes of trimedial quasigroups (cf. 1.4).

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