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AN EXISTENCE THEOREM FOR THE URYSOHN INTEGRAL EQUATION IN BANACH SPACES

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Abstract: The paper contains an existence theorem for L_φ-solutions of the Urysohn integral equation, where $L_\varphi(D,X)$ is a generalized Orlicz space over a Banach space X. For the case when X is finite dimensional and φ is a usual N-function, our theorem reduces to some results from Ch. IV of [4].

Key words: Urysohn integral equations, Orlicz spaces, measure of non-compactness.

Classification: 45N05

Let X be a separable Banach space and let D be a compact subset of the Euclidean space R^m. In this paper we shall present sufficient conditions for the existence of a solution x of the integral equation

(1) $$x(t) = p(t) + \lambda \int_D f(t,s,x(s))ds$$

belonging to a certain Orlicz space $L_\varphi(D,X)$.

1. Preliminaries. A function $\varphi : R_+ \times D \to R_+$ is called a (generalized) N-function if

(i) $\varphi(0,t) = 0$ for almost all $t \in D$;

(ii) for almost every $t \in D$ the function $u \to \varphi(u,t)$ is convex and non-decreasing on R_+;

(iii) for any $u \in R_+$ the function $t \to \varphi(u,t)$ is L-measurable on D;

(iv) for almost every $t \in D$

$$\lim_{u \to 0} \frac{\varphi(u,t)}{u} = 0 \text{ and } \lim_{u \to 0} \frac{\varphi(u,t)}{u} = \infty .$$

The function φ^* defined by

$$\varphi^*(u,t) = \sup_{v \geq 0} (uv - \varphi(v,t)) \quad (u \geq 0,\ t \in D)$$

is called the complementary function to φ.

For a given N-function φ we denote by $L_\varphi(D,R)$ the set of all L-measurable functions $u: D \to R$ for which the number

$$\|u\|_\varphi = \inf\{r > 0: \int_D \varphi(|u(t)|/r, t)dt \leq 1\}$$

is finite. $L_\varphi(D,R)$ is called the (generalized) Orlicz space. It is well known (cf. [3],[4]) that $\langle L_\varphi(D,R), \|\cdot\|_\varphi\rangle$ is a Banach space and

1.1. The convergence in $L_\varphi(D,R)$ implies the convergence in measure.

1.2. For any $u \in L_\varphi(D,R)$ and $v \in L_{\varphi^*}(D,R)$ the function uv is integrable and

$$\int_D |u(t)v(t)|dt \leq 2\|u\|_\varphi \|v\|_{\varphi^*} \quad \text{(Hölder's inequality).}$$

If, in addition, the function φ satisfies Condition A:

$$\int_D \varphi(u,t)dt < \infty \quad \text{for all } u > 0,$$

then we may consider the set $E_\varphi(D,R)$ defined to be the closure in $L_\varphi(D,R)$ of the set of simple functions. Clearly $E_\varphi(D,R)$ is a Banach subspace of $L_\varphi(D,R)$. It can be shown (cf. [3],[4]) that

1.3. The following statements are equivalent:

(i) $x \in E_\varphi(D,R)$;

(ii) $x \in L_\varphi(D,R)$ and x has absolutely continuous norm;

(iii) $\int_D \varphi(\alpha|u(t)|, t)dt < \infty$ for all $\alpha > 0$.

1.4. If a sequence (u_n) in $E_\varphi(D,R)$ has equi-absolutely continuous norms and converges in measure, then (u_n) converges in $E_\varphi(D,R)$.

Further, denote by $L_\varphi(D,X)$ the set of all strongly measurable functions $x:D \to X$ such that $\|x\| \in L_\varphi(D,R)$. Analogously we define $E_\varphi(D,X)$. Then $L_\varphi(D,X)$ is a Banach space with the norm $\|x\|_\varphi = \| \, \|x\| \, \|_\varphi$. Moreover, let $L^1(D,X)$ denote the Lebesgue space of all (Bochner) integrable functions $x:D \to X$ provided with the norm $\|x\|_1 = \int_D \|x(t)\| dt$. We shall always assume that all functions from $L^1(D,X)$ are extended to R^m by putting $x(t) = 0$ for $t \in R^m \setminus D$.

Let β and β_1 be the Hausdorff measures of noncompactness (cf. [6]) in X and $L^1(D,X)$, respectively. For any set V of functions from D into X denote by v the function defined by $v(t) = \beta(V(t))$ for $t \in D$ (under the convention that $\beta(A) = \infty$ if A is unbounded), where $V(t) = \{x(t) : x \in V\}$. In what follows we shall use the following

<u>Theorem 1</u>. Let V be a countable subset of $L^1(D,X)$ such that there exists $\mu \in L^1(D,R)$ such that $\|x(t)\| \le \mu(t)$ for all $x \in V$ and $t \in D$. Then the function v is integrable on D and for any measurable subset T of D

$$(2) \qquad \beta(\{\int_T x(t)dt : x \in V\}) \le \int_T v(t)dt.$$

If, in addition, $\lim_{\tau \to 0} \sup_{x \in V} \int_D \|x(t+\tau) - x(t)\| dt = 0$, then

$$\beta_1(V) \le \int_D v(t)dt.$$

We omit the proof of this theorem, because it is similar to that of Theorem 1 from [5].

2. <u>The main result</u>. Assume now that

1° $M,N:R_{+}\times D \to R_{+}$ are complementary N-functions and M satisfies Condition A.

2° $\varphi:R_{+}\times D \to R_{+}$ is an N-function satisfying Condition A and such that

(3) $\quad u \leq c\varphi(u,t) + a(t)$ for all $u \geq 0$ and a.a. $t \in D$,

where c is a positive number and $a \in L^{1}(D,R)$. Let ψ be the complementary function to φ.

3° $(t,s,x) \to f(t,s,x)$ is a function from $D^{2}\times X$ into X which is continuous in x for a.e. $t,s \in D$ and strongly measurable in (t,s) for every $x \in X$.

4° $\|f(t,s,x)\| \leq K(t,s)g(s,\|x\|)$ for $t,s \in D$ and $x \in X$, where

(i) $(s,u) \to g(s,u)$ is a function from $D\times R_{+}$ into R_{+}, measurable in s and continuous in u, and there exist $\alpha, \gamma > 0$ and $b \in L^{1}(D,R)$, $b \geq 0$, such that $N(\alpha g(s,u),s) \leq \gamma\varphi(u,s) + b(s)$ for all $u \geq 0$ and a.a. $s \in D$;

(ii) $(t,s) \to K(t,s)$ is a function from D^{2} into R_{+} such that $K(t,\cdot) \in E_{M}(D,R)$ for a.e. $t \in D$ and the function $t \to \|K(t,\cdot)\|_{M}$ belongs to $E_{\varphi}(D,R)$.

For simplicity put $L^{1} = L^{1}(D,X)$, $L_{\varphi} = L_{\varphi}(D,X)$, $E_{\varphi} = E_{\varphi}(D,X)$ and $B_{\varphi}^{r} = \{x \in E_{\varphi}: \|x\|_{\varphi} \leq r\}$. Let F be the mapping defined by

$$F(x)(t) = \int_{D} f(t,s,x(s))ds \quad (x \in E_{\varphi},\ t \in D).$$

<u>Theorem 2</u>. Assume in addition that

5° $\lim\limits_{\tau\to 0}\ \sup\limits_{x\in B_{\varphi}^{r}} \int_{D} \|F(x)(t+\tau) - F(x)(t)\|\,dt = 0$ for all $r > 0$

and

6° $\beta(f(t,s,Z)) \leq H(t,s)\beta(Z)$ for almost every $t,s \in D$ and for every bounded subset Z of X, where $(t,s) \to H(t,s)$ is a function from D^2 into R_+ such that $H(t,\cdot) \in L_\Psi(D,R)$ for a.e. $t \in D$ and the function $t \longrightarrow \|H(t,\cdot)\|_\Psi$ belongs to $L_\Phi(D,R)$.

Then for any $p \in E_\varphi$ there exists a positive number ρ such that for any $\lambda \in R$ with $|\lambda| < \rho$ the equation (1) has a solution $x \in E_\varphi$.

<u>Remark 1</u>. For example, the condition 5^0 holds if

$$f(t,s,x) = K(t,s)q(s,x)$$

and $\lim_{\tau \to 0} \int_D \|K(t+\tau,\cdot) - K(t,\cdot)\|_M dt = 0$ and $\|q(s,x)\| \leq g(s, \|x\|)$ for $x \in X$ and a.e. $s \in D$.

<u>Remark 2</u>. The condition 6^0 holds whenever $f = f_1 + f_2$, where f_1 and f_2 are such that

($*$) for a.e. $t,s \in D$ the function $x \longrightarrow f_1(t,s,x)$ is completely continuous;

($**$) $\|f_2(t,s,x) - f_2(t,s,y)\| \leq H(t,s)\|x - y\|$ for $x,y \in X$ and a.e. $t,s \in D$.

<u>Proof</u>. By 4^0 and the Hölder inequality we have

$$\|F(x)(t)\| \leq 2\|K(t,\cdot)\|_M \|g(\cdot,\|x\|)\|_N \text{ for } t \in D.$$

Since

$$\|g(\cdot,\|x\|)\|_N = \frac{1}{\alpha}\|\alpha g(\cdot,\|x\|)\|_N \leq \frac{1}{\alpha}(1 + \int_D N(\alpha g(s,\|x(s)\|),s)ds) \leq$$

$$\frac{1}{\alpha}(1 + \int_D b(s)ds + \gamma \int_D \varphi(\|x(s)\|,s)ds),$$

we get

(4) $\|F(x)(t)\| \leq k(t)(1 + \|b\|_1 + \gamma r_\varphi(x))$ for $x \in E_\varphi$ and $t \in D$,

where $k(t) = \frac{2}{\alpha}\|K(t,\cdot)\|_M$ and $r_\varphi(x) = \int_D \varphi(\|x(s)\|,s)ds$. From 4^0 (ii) and (3) it is clear that $k \in E_\varphi(D,R) \cap L^1(D,R)$. Hence

(5) $\|F(x)\chi_T\|_\varphi \leq \|k\chi_T\|_\varphi (1 + \|b\|_1 + \gamma r_\varphi(x))$

for $x \in E_\varphi$ and any measurable subset T of D.

Similarly it can be shown that

(6) $\int_T \|f(t,s,x(s))\| \, ds \le \frac{2}{\alpha} \|K(t,\cdot)\chi_T\|_M (1 + \|b\|_1 + \gamma r_\varphi(x))$

for $x \in E_\varphi$, $t \in D$ and any measurable subset T of D.

In virtue of 1.3, from (5) we infer that F is a mapping of E_φ into itself. We shall show that F is continuous. Let x_n, $x_0 \in E_\varphi$ and $\lim_{n\to\infty} \|x_n - x_0\|_\varphi = 0$. Suppose that $\|F(x_n) - F(x_0)\|_\varphi$ does not converge to 0 as $n \to \infty$. Thus there exist $\varepsilon > 0$ and a subsequence (x_{n_j}) such that

(7) $\|F(x_{n_j}) - F(x_0)\|_\varphi > \varepsilon$ for $j = 1,2,\ldots$

and $\lim_{j\to\infty} x_{n_j}(t) = x_0(t)$ for a.e. $t \in D$. From 1.3 and the inequality

$$r_\varphi(x_n) \le \frac{1}{2} r_\varphi(2(x_n - x_0)) + \frac{1}{2} r_\varphi(2x_0)$$

it follows the boundedness of the sequence $(r_\varphi(x_n))$. By (6) this implies that for a.e. $t \in D$ the sequence $(\|f(t,s,x_n(s))\|)$ is equi-integrable on D. As $\lim_{j\to\infty} f(t,s,x_{n_j}(s)) = f(t,s,x_0(s))$ for a.e. $t,s \in D$, the Vitali convergence theorem proves that

$$\lim_{j\to\infty} F(x_{n_j})(t) = F(x_0)(t) \quad \text{for a.e. } t \in D.$$

Moreover, in view of (5), the sequence $(F(x_{n_j}))$ has equi-absolutely continuous norms in L_φ. Thus, by 1.4, $\lim_{j\to\infty} \|F(x_{n_j}) - F(x_0)\|_\varphi = 0$ which contradicts (7).

Fix a function $p \in E_\varphi$. Denote by Q the set of all $q > 0$ for which there exists $r > 0$ such that $\int_D \varphi(\|p(t)\| + qk(t)(1 + \|b\|_1 + \gamma r), t)dt \le r$. Let $\rho = \min(\sup Q, 1/\|h\|_\varphi)$, where $h(t) = \|H(t,\cdot)\|_\psi$ for $t \in D$.

Fix $\lambda \in R$ with $|\lambda| < \rho$. From the definition of ρ we deduce that there exists $d>0$ such that

(8) $\int_D \varphi(\|p(t)\| + |\lambda| k(t)(1 + \|b\|_1 + \gamma d), t)dt \le d.$

Set $U = \{x \in E_\varphi : r_\varphi(x) \le d\}$ and $G(x) = p + \lambda F(x)$ for $x \in E_\varphi$. Then G is a continuous mapping $E_\varphi \to E_\varphi$ and, by (4) and (8), $G(U \subset U)$. Consequently

(9) $$G(\overline{U}) \subset \overline{G(U)} \subset \overline{U}.$$

Obviously, $\overline{U}$ is a bounded, closed and convex subset of E_φ, and

(10) $$\overline{U} \subset B_\varphi^{d+1}.$$

Now we shall show that for any countable subset V of $\overline{U}$

(11) $V \subset \overline{\mathrm{conv}}\,(G(V) \cup \{0\}) \implies V$ is relatively compact in E_φ.

Assume that V is a countable set of functions belonging to $\overline{U}$ and

(12) $$V \subset \overline{\mathrm{conv}}\,(G(V) \cup \{0\}).$$

Owing to 1.1 it is clear that

$$V(t) \subset \overline{\mathrm{conv}\,(G(V)(t) \cup \{0\})} \text{ for a.e. } t \in D,$$

so that

(13) $$\beta(V(t)) \le \beta(G(V)(t)) \text{ for a.e. } t \in D.$$

From (4) it follows that for any $y \in \overline{G(U)}$

$$\|y(t)\| \le \mu(t) \quad \text{for a.e. } t \in D,$$

where $\mu(t) = \|p(t)\| + |\lambda| k(t)(1 + \|b\|_1 + \gamma d)$. As V is countable, in view of (9) and (12), this implies that there exists a set D_0 of Lebesgue measure zero such that

(14) $$\|x(t)\| \le \mu(t) \quad \text{for all } x \in V \text{ and } t \in D \setminus D_0.$$

Let us remark that $\mu \in E_\varphi(D,R) \cap L^1(D,R)$.

On the other hand, by 5°, (10) and (12), we have

$$\lim_{\tau \to 0} \sup_{x \in V} \int_D \|x(t+\tau) - x(t)\| dt = 0.$$

Hence, by Theorem 1, the function $t \to v(t) = \beta(V(t))$ is integrable on D and

(15) $$\beta_1(V) \leq \int_D v(t)dt.$$

Furthermore, from 4° and (14) it follows that for any $t \in D$ such that $K(t,\cdot) \in E_M(D,R)$, we have

$$\| f(t,s,x(s)) \| \leq \eta(s) \quad \text{for } x \in V \text{ and a.e. } s \in D,$$

where $\eta(s) = K(t,s)g(s,\mu(s))$. As $\mu \in E_\varphi(D,R)$, 4°(i) implies that $g(\cdot,\mu) \in L_N(D,R)$, and consequently, by the Hölder inequality, $\eta \in L^1(D,R)$. Hence, owing to 6° and (2),

$$\beta(G(V)(t)) = \beta(\{\lambda \int_D f(t,s,x(s))ds : x \in V\}) \leq$$

$$|\lambda| \int_D \beta(\{f(t,s,x(s)) : x \in V\})ds \leq |\lambda| \int_D H(t,s)\,\beta(V(s))ds$$

In view of (13), this shows that

$$v(t) \leq |\lambda| \int_D H(t,s)v(s)ds \quad \text{for a.e. } t \in D.$$

Moreover, by (14), we have $v(t) \leq \mu(t)$ for a.e. $t \in D$, and therefore $v \in E_\varphi(D,R)$. Thus, by the Hölder inequality,

$$v(t) \leq |\lambda| \, \| H(t,\cdot) \|_\psi \, \| v \|_\varphi \quad \text{for a.e. } t \in D,$$

so that

$$\| v \|_\varphi \leq |\lambda| \, \| h \|_\varphi \, \| v \|_\varphi .$$

Since $|\lambda| \, \| h \|_\varphi < 1$, this implies that $\| v \|_\varphi = 0$, i.e. $v(t) = 0$ for a.e. $t \in D$. Hence, by (15), $\beta_1(V) = 0$, i.e. V is relatively compact in L^1. On the other hand, as $\mu \in E_\varphi(D,R)$, (14) implies that V has equi-absolutely continuous norms in L_φ. From this we deduce that V is relatively compact in E_φ, which proves (11).
Applying now Daher's generalization of the Schauder fixed point theorem (cf. [1]), we conclude that there exists $x \in \bar{U}$ such that $x = G(x)$. It is clear that x is a solution of (1).

R e f e r e n c e s

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