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CONSTRUCTION OF MEDIAL SEMIGROUPS

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Abstract: Every medial semigroup, satisfying a certain condition, is a subsemigroup of a medial semigroup, which is constructed by means of commutative semigroups and their commuting and idempotent endomorphisms.

Key words: Semigroup, medial semigroup, endomorphism.

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Let (H,+) be a commutative semigroup and φ, ψ its idempotent permutable endomorphisms, $\varphi^2 = \varphi$, $\psi^2 = \psi$, $\varphi\psi = \psi\varphi$. By the definition

(1) $$ab = \varphi(a) + \psi(b)$$

we obtain a medial semigroup (H,), that is, a semigroup satisfying the identity uvxy = uxvy. Moreover (H,·) is satisfying the implication

(✱) $ab = cd \Longrightarrow axb = cxd$ for all $a,b,c,d,x \in H$ (see [4]).

It is easy to see that not every medial semigroup with (✱) can be constructed in this way ([4]). In the case of groupoids there are several theorems giving conditions for a medial groupoid to be constructable from commutative groupoids by the definition (1) ([1],[2],[3],[5]). We prove in this note that every medial semigroup, satisfying (✱), is a subsemigroup of a medial semigroup (H,·), obtained from a commutative semigroup (H,+)

by its idempotent permutable endomorphisms φ and ψ , where the multiplication is defined by (1).

1. <u>The lemmas.</u> Let $(\{x_i\},\ i \in I; \cdot)$ be a medial semigroup satisfying $(*)$. Let $\sim_l$, $\sim_r$, $\sim_t$ be the following relations

$$x_j \sim_l x_k \Longleftrightarrow \begin{cases} x_j = x_k \text{ or there are } y_1,\dots,y_n \in X,\ n>1, \text{ and a} \\ \text{permutation } \pi \text{ of the numbers } 2,3,\dots,n \text{ with} \\ x_j = y_1,\dots,y_n \text{ and } x_k = y_1 y_{\pi(2)},\dots,y_{\pi(n)}. \end{cases}$$

$$x_j \sim_r x_k \Longleftrightarrow \begin{cases} x_j = x_k \text{ or there are } y_1,\dots,y_n \in X,\ n>1, \text{ and a} \\ \text{permutation } \pi \text{ of the numbers } 1,\dots,n-1 \text{ with} \\ x_j = y_1,\dots,y_n,\ x_k = y_{\pi(1)},\dots,y_{\pi(n-1)} y_n. \end{cases}$$

$$x_j \sim_t x_k \Longleftrightarrow \begin{cases} \text{There are } y_1,\dots,y_n \in X,\ n \geq 1, \text{ and a permutation} \\ \pi \text{ of the numbers } 1,2,\dots,n \text{ with} \\ x_j = y_1,\dots,y_n,\ x_k = y_{\pi(1)},\dots,y_{\pi(n)}. \end{cases}$$

<u>Lemma 1.</u> a) The relations $\sim_l$, $\sim_r$, $\sim_t$ are reflexive, symmetric and stable with respect to the multiplication.

b) $\sim_r \subseteq \sim_t$ and $\sim_l \subseteq \sim_t$.

c) $x_j \sim_l x_k \Longrightarrow x_j y = x_k y$ for all $y \in X$,

$x_j \sim_r x_k \Longrightarrow y x_j = y x_k$ for all $y \in X$,

$x_j \sim_t x_k \Longrightarrow x_j y \sim_r x_k y$ and $y x_j \sim_l y x_k$ for all $y \in X$.

The transitive closures of $\sim_l$, $\sim_r$, $\sim_t$ are congruences and we denote the congruence classes containing y by $[y]_l$, $[y]_r$ and $[y]_t$ respectively.

<u>Lemma 2.</u> From $[y_1]_l = [y_2]_l$ and $[x_1]_t = [x_2]_t$ it follows: $[y_1 x_1]_l + [y_2 x_2]_l$. From $[y_1]_r = [y_2]_r$ and $[x_1]_t = [x_2]_t$ it follows $[x_1 y_1]_r = [x_2 y_2]_r$. From $[y_1]_l = [y_2]_l$ and $[x_1]_r = [x_2]_r$ it follows $y_1 x_1 = y_2 x_2$.

For a given medial semigroup $(X,\cdot) = (\{x_i\}, i\in I;\cdot)$ let $F = (F,+) = F(a_i,b_i,c_i,d_i)$, $i\in I$, be the free commutative semigroup with the free system of generators $\{a_i\}\cup\{b_i\}\cup\{c_i\}\cup\{d_i\}$, $i\in I$ $(F\cap X = \emptyset)$. We denote the elements R, S of F by formal infinite sums

$R = \sum \alpha_i a_i + \beta_i b_i + \gamma_i c_i + \delta_i d_i,\ \ \sum \alpha_i + \beta_i + \gamma_i + \delta_i \geq 1.$

Let $\sim$ be the following relation on F

(1) $R\sim S \Longleftrightarrow R = S$ or

(2) $R = \sum \delta_i d_i$ and $S = \sum \delta'_i d_i$ with $[\prod x_i^{\delta_i}]_t = [\prod x_i^{\delta'_i}]_t$ or

(3) $R = b_h + \sum^* \delta_i d_i$, $S = b_j + \sum^* \delta'_i d_i$ with

$[x_h \prod^* x_i^{\delta_i}]_1 = [x_j \prod^* x_i^{\delta'_i}]_1$ or

(4) $R = c_h + \sum^* \delta_i d_i$, $S = c_j + \sum^* \delta'_i d_i$ with

$[(\prod^* x_i^{\delta_i})x_h]_r = [(\prod^* x_i^{\delta'_i})x_j]_r$ or

(5) $R = b_h + c_k + \sum^* \delta_i d_i$, $S = b_j + c_m + \sum^* \delta'_i d_i$ with

$x_h \prod^* x_i^{\delta_i} x_k = x_j \prod^* x_i^{\delta'_i} x_m.$

By the starlet at the sums or products we denote the possibility of being empty.

Lemma 3. Let $\sum \mu_i d_i \sim \sum \mu'_i d_i$.

a) If $R\sim S$ according to (2), then $\sum(\delta_i + \mu_i)d_i \sim \sum(\delta'_i + \mu'_i)d_i$.

b) If $R\sim S$ according to (3), then

$b_h + \sum^*(\delta_i + \mu_i)d_i \sim b_j + \sum^*(\delta'_i + \mu'_i)d_i.$

c) If $R\sim S$ according to (4), then

$c_h + \sum^*(\delta_i + \mu_i)d_i \sim c_j + \sum^*(\delta'_i + \mu'_i)d_i.$

d) If $R\sim S$ according to (5), then

$b_h + c_k + \sum^*(\delta_i + \mu_i)d_i \sim b_j + c_m + \sum^*(\delta'_i + \mu'_i)d_i.$

Proof. We know $[\prod x_i^{\mu_i}]_t = [\prod x_i^{\mu'_i}]_t$.

a) We have $[\prod x_i^{\delta_i}]_t = [\prod x_i^{\delta'_i}]_t$ and therefore $[\prod x_i^{\delta_i}]_t [\prod x_i^{\mu_i}]_t = [\prod x_i^{\delta'_i}]_t [\prod x_i^{\mu'_i}]_t = [\prod x_i^{\delta_i+\mu_i}]_t = [\prod x_i^{\delta'_i+\mu'_i}]_t$.

b) Using Lemma 2 from $[x_h \prod^* x_i^{\delta_i}]_1 = [x_j \prod^* x_i^{\delta'_i}]_1$ we have $[x_j \prod^* x_i^{\delta_i+\mu_i}]_1 = [x_j \prod^* x_i^{\delta'_i+\mu'_i}]_1$.

c) analogous to b)

d) We have $x_h \prod^* x_i^{\delta_i} x_k = x_j \prod^* x_i^{\delta'_i} x_m$. With respect to the condition (✱) it follows $x_h \prod^* x_i^{\delta_i} \prod^* x_i^{\mu_i} x_k = x_j \prod^* x_i^{\delta'_i} x_i^{\mu_i} x_m$. This is by Lemma 1 equal to $x_j \prod^* x_i^{\delta'_i} \prod^* x_i^{\mu'_i} x_m$ and because of the mediality of X we have $x_h \prod^* x_i^{\delta_i+\mu_i} x_k = x_j \prod^* x_i^{\delta'_i+\mu'_i} x_m$. Using the relation $\sim$ we define a relation $\triangle$ on $(F,+)$:

$R = \sum \alpha_i a_i + \beta_i b_i + \gamma_i c_i + \delta_i d_i \triangle S = \sum \alpha'_i a_i + \beta'_i b_i + \gamma'_i c_i + \delta'_i d_i$ iff there exist $A_1,\dots,A_n, A'_1,\dots,A'_n \in (F,+)$ with $R = \sum_{i=1}^{m} A_i$, $S = \sum_{i=1}^{m} A'_i$ and $A_i \sim A_i$.

The relation $\triangle$ is reflexive, symmetric and stable with respect to addition. The transitive closure $\equiv$ is a congruence on $(F,+)$. By $[R] = [\sum \alpha_i a_i + \beta_i b_i + \gamma_i c_i + \delta_i d_i]$ we denote the class containing $R = \sum \alpha_i a_i + \beta_i b_i + \gamma_i c_i + \delta_i d_i$.

Lemma 4. a) $[a] = \{a\}$.

b) If $a = b_m + c_n + \sum^* \delta_i d_i$ and $A \equiv B$, then $A \sim B$ follows.

Proof. a) Since a is an element of the free system of generators of F, from $a \triangleq R$ it follows $a \sim R$ and therefore $a = R$.

b) Let $A \triangleq B$; then there exist $A_1,\dots,A_n$, $A'_1,\dots,A'_n$ with $A = \Sigma A_j$, $B = \Sigma A'_j$ and $A_j \sim A'_j$ for $j = 1,\dots,n$. b_m and c_n are elements of the free system of generators and therefore only the following two cases are possible.

1) One of the elements A_j, say A_1, is of the form $A_1 = b_m + c_n + \Sigma^* \mu_i d_i$. It follows $A'_1 = b_k + c_h + \Sigma^* \mu'_i d_i$ and all other elements A_i are of the form $\Sigma \lambda_i d_i$. In view of Lemma 3 we can write $A = A_1 + A_2$, $B = A'_1 + A'_2$, where $A_2 = \Sigma \varkappa_i d_i$, $A'_2 = \Sigma \varkappa'_i d_i$ and $A_2 \sim A'_2$. Again in view of Lemma 3 we get $A \sim B$.

2) One of the elements A_j, say A_1, is of the form $A_1 = b_m + \Sigma^* \mu_i d_i$, another, say A_2, of the form $A_2 = c_n + \Sigma^* \varkappa_i d_i$. Then $A'_1 = b_k + \Sigma^* \mu'_i d_i$ and $A'_2 + c_h + \Sigma^* \varkappa'_i d_i$. In view of Lemma 3 we may write $A = A_1 + A_2 + A_3$, $B = A'_1 + A'_2 + A'_3$, where $A_3 = \Sigma^* \lambda_i d_i$, $A'_3 = \Sigma^* \lambda'_i d_i$ and $A_3 \sim A'_3$. We have

$$[x_m \Pi^* x_i^{\mu_i}]_1 = [x_k \Pi^* x_i^{\mu'_i}]_1,\ [\Pi^* x_i^{\,i}\, x_n]_r = [\Pi^* x_i^{\,i}\, x_n]_r$$

and $[\Pi^* x_i^{\lambda_i}]_t = [\Pi^* x_i^{\lambda'_i}]_t$. Lemma 1 gives $[x_m \Pi^* x_i^{\mu_i} \Pi^* x_i^{\lambda_i}]_1 = [x_k \Pi^* x_i^{\mu'_i} \Pi^* x_i^{\lambda'_i}]_1$ and Lemma 2 the equality

$$x_m \Pi^* x_i^{\mu_i} \Pi^* x_i^{\lambda_i} \Pi^* x_i^{\varkappa_i} x_n = x_k \Pi^* x_i^{\mu'_i} \Pi^* x_i^{\lambda'_i} \Pi^* x_i^{\varkappa_i} x_h,$$

therefore we have $A \sim B$.

The relation $\sim$ is transitive, therefore from $A \equiv B$ it follows $A \sim B$.

2. The theorems. We define homomorphisms φ_0 and ψ_0 from $(F,+)$ into $(F,+)$ by

(6) $\varphi_o(a_i) = b_i$, $\varphi_o(b_i) = b_i$, $\varphi_o(c_i) = d_i$, $\varphi_o(d_i) = d_i$

(7) $\psi_o(a_i) = c_i$, $\psi_o(b_i) = d_i$, $\psi_o(c_i) = c_i$, $\psi_o(d_i) = d_i$.

Lemma 5. a) The endomorphisms φ_o and ψ_o are idempotent and permutable, $\varphi_o^2 = \varphi_o$, $\psi_o^2 = \psi_o$ and $\varphi_o\psi_o = \psi_o\varphi_o$.

b) $R \equiv S$ implies $\varphi_o(R) \equiv \varphi_o(S)$ and $\psi_o(R) \equiv \psi_o(D)$.

Proof. a) Easy, since the conditions are satisfied for the system of free generators.

b) It suffices to prove that $R \sim S$ implies $\varphi_o(R) \sim \varphi_o(S)$ and $\psi_o(R) \sim \psi_o(S)$. This is clear for the cases 1) $R = S$ and 2) $R = \Sigma\, \sigma_i d_i$, $S = \Sigma\, \sigma'_i d_i$. Let $R \sim S$ according to (3). Then we have $\varphi_o(R) = R$, $\varphi_o(S) = S$, $\psi_o(R) = d_h + \Sigma^* \sigma_i d_i$, $\psi_o(S) = d_j + \Sigma^* \sigma'_i d_i$. In view of Lemma 1 we have $\sim_{\tau} \subseteq \sim_t$ and thus $\psi_o(R) \sim \psi_o(S)$. Analogously we prove the case 4), $R \sim S$ according to (4). Case 5, let $R \sim S$ according to (5), hence $x_h \Pi^* x_i^{\sigma_i} x_k = x_j \Pi^* x_i^{\sigma'_i} x_m$. We have $\varphi_o(R) = b_h + d_k + \Sigma^* \sigma_i d_i$, $\varphi_o(S) = b_j + d_m + \Sigma^* \sigma'_i d_i$. The relation $\varphi_o(R) \sim \varphi_o(S)$ follows from $[x_h x_k \Pi^* x_i^{\sigma_i}]_1 = [x_h \Pi^* x_i^{\sigma_i} x_k]_1 = [x_j \Pi^* x_i^{\sigma'_i} x_m]_1 = [x_j x_m \Pi^* x_i^{\sigma'_i}]_1$. $\psi_o(R) \sim \psi_o(S)$ follows analogously.

We know by Lemma 5b) that the endomorphisms φ_o and ψ_o induce endomorphisms φ and ψ of $F/\equiv$, satisfying again the condition (6) and (7). From this we have by an easy calculation:

Theorem 1 (see [4]). $F/\equiv$ is a medial semigroup with respect to the multiplication $[R][S] = [\varphi(R)] + [\psi(S)]$.

If $x_h \Pi^* x_i^{\sigma_i} x_k = x_n$, then we denote the class $[b_h + c_k + \Sigma^* \sigma_i d_i$ by T_n. By the lemmas 4 and 5, this notation does not depend on the chocie of the representatives.

Theorem 2. a) The set

$$T = \{[a_j] : x_j \notin X^2\} \cup (\cup T_n)$$

is a medial subsemigroup of $(F/\equiv, \cdot)$

b) The mapping $\wp$

$$\wp(x_j) = \begin{cases} [a_j] \text{ if } x_j \notin X^2 \\ T_j \text{ if } x_j \in X^2 \end{cases}$$

is an isomorphism from X onto $T \subseteq F/\equiv$

Proof. a) Let x_j and $x_k \notin X^2$. We have $[a_j][a_k] = \varphi(a_j) + \psi(a_k) = b_j + c_k$ and thus this expression is of the form (5). Let $x_j \notin X^2$, $T_n = [b_1 + c_k + \Sigma^* \sigma_i d_i]$. We have $[a_j] T_n = \varphi(a_j) + \psi(b_1 + c_k + \Sigma^* \sigma_i d_i) = b_j + d_1 + c_k + \Sigma^* \sigma_i d_i$ and thus this expression is of the form (5). We have $T_n [a_j] = \varphi(b_1 + c_k + \Sigma^* \sigma_i d_i) + \psi(a_j) = b_1 + d_k + \Sigma^* \sigma_i d_i + c_j$ and thus this expression is of the form (5). Further we have $[b_1 + c_k + \Sigma^* \sigma_i d_i][b_j + c_m + \Sigma^* \sigma'_i d_i] = b_1 + d_k + \Sigma^* \sigma_i d_i + d_j + c_m + \Sigma^* \sigma_i d_i$ and this expression is of the form (5). T is a subsemigroup of $F/\equiv$.

b) $\wp$ is a bijection. Let $x_r x_s = x_t$.

b1) Let $x_r, x_s \notin X^2$. We have $\wp(x_r) = [a_r]$, $\wp(x_s) = [a_s]$, $[a_r][a_s] = [\varphi(a_r) + \psi(a_s)] = [b_r + c_s] = T_t$.

b2) Let $x_r \notin X^2$, $x_s \in X^2$. We have $\wp(x_r) = [a_r]$ and $\wp(x_s) = T_s = [b_1 + c_k + \Sigma^* \sigma_i d_i]$, with $x_1 \Pi^* x_i^{\sigma_i} x_k = x_s$. It holds $\wp(x_r)\wp(x_s) = [a_j] T_s = [b_r] + [\psi(T_s)] = [b_r + d_1 + c_k + \Sigma^* \sigma_i d_i] = T_t$, because of $x_r x_1 \Pi^* x_i^{\sigma_i} x_k = x_r x_s = x_t$.

b3) Let $x_r \in X^2$, $x_s \notin X^2$. We have $\wp(x_r) = T_r = [b_1 + c_k + \Sigma^* \sigma_i d_i]$, where $x_1 \Pi^* x_i^{\sigma_i} x_k = x_r$. It holds $\wp(x_r)\wp(x_s) = T_r [a_s] = [b_1 + d_k + \Sigma^* \sigma_i d_i] + c_s = T_t = \wp(x_r x_s)$, because of $x_1 x_k \Pi^* x_i^{\sigma_i} x_s = x_r x_s = x_t$.

b4) Let $x_r, x_j \in X^2$. We have $\wp(x_r) = T_r = [b_1 + c_k + \Sigma^* \sigma_i d_i]$, $\wp(x_s) = T_s = [b_j + c_m + \Sigma^* \sigma_i d_i$ with $x_1 \Pi^* x_i^{\sigma_i} x_k = x_r$, $x_j \Pi^* x_i^{\sigma_i} x_m = x_s$. Hence, $\wp(x_r)\wp(x_s) = T_r T_s = \varphi(T_r) + \psi(T_s) = [b_1 + d_k + \Sigma^* \sigma_i d_i + d_j + c_m + \Sigma^* \sigma_i d_i] = T_t = \wp(x_r x_s)$.

<u>Theorem 3</u>. Let $X = \{x_i, i \in I\}$ be a medial and archimedean semigroup. Then $(F/\equiv, \cdot)$ is archimedean, too.

Proof. Let $A = \Sigma\, \alpha_i a_i + \beta_i b_i + \gamma_i c_i + \sigma_i d_i$ and $B = \Sigma\, \varkappa_i a_i + \lambda_i b_i + \mu_i c_i + \nu_i d_i$. Since X is archimedean, there exist a natural number $n \geq 1$ and elements x_r and x_s with $x_r \Pi\, x_i^{\alpha_i+\beta_i+\gamma_i+\sigma_i} x_s = (\Pi\, x_i^{\varkappa_i+\lambda_i+\mu_i+\nu_i})^n$. Therefore we have $\varphi_o \psi_o(d_r A d_s) = d_r + \Sigma(\alpha_i + \beta_i + \gamma_i + \sigma_i) d_i + d_s \sim n\, \Sigma(\varkappa_i + \lambda_i + \mu_i + \nu_i) d_i = \varphi_o \psi_o(B^n)$. From this $Bd_r A d_s B = B^{n+2}$ and consequently $(F/\equiv, \cdot)$ is an archimedean semigroup.

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