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ON BOUNDED SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION WITH A NONLINEAR PERTURBATION

Bogdan RZEPECKI

Abstract: Let E be a Banach space. Suppose that $f:[0,\infty)\times E \rightarrow E$ satisfies the Carathéodory conditions and some regularity condition expressed in terms of the measure of noncompactness α. We prove the existence of bounded solutions of the differential equation $y' = A(t)y + f(t,y)$ under the assumption that the linear equation $y' = A(t)y + b(t)$ has at least one bounded solution for each b belonging to a function Banach space B_0.

Key words: Differential equations in Banach spaces, function spaces, admissibility, measure of noncompactness.

Classification: 34G20, 34A34, 34C11.

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1. Introduction. Throughout this paper, J denotes the half-line $t \geq 0$, E a Banach space with the norm $\|\cdot\|$, and $\mathcal{L}(E)$ the algebra of continuous linear operators from E into itself with the induced standard norm $|\cdot|$.

Consider the nonlinear differential equation

(+) $$y'(t) = A(t)y(t) + f(t,y(t)),$$

where $t \in J$, $A(t) \in \mathcal{L}(E)$, and f is an E-valued function defined on $J \times E$.

We are interested in the study of bounded solutions of (+) when f satisfies the Carathéodory conditions and some regularity Ambrosetti-Szufla type condition (cf. [1],[11]) expressed in terms of the measure of noncompactness α.

The method used here is based on the concept of "admissibility" due to Massera and Schäffer [8]. With (+) above we shall associate the nonhomogeneous linear equation

$$(*) \qquad y'(t) = A(t)y(t) + b(t)$$

under the assumption that has at least one bounded solution for each function b belonging to a function Banach space B_0.

2. Notation and preliminaries. Let α denote the Kuratowski's measure of noncompactness in E. (The measure $\alpha(X)$ of a nonempty bounded subset X of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leq \varepsilon$.) For properties of the Kuratowski function α, see e.g. [3] - [6],[10].

Further, we will use the standard notations. The closure of a set X, its diameter and its closed convex hull be denoted, respectively, by $\overline{X}$, diam X and $\overline{\text{conv}}$ X. If X and Y are subsets of E and t, s are real numbers, then tX + sY is the set of all tx + sy such that $x \in X$ and $y \in Y$. For a set $\mathcal{V}$ of mappings defined on X we write $\mathcal{V}(t) = \{\varphi(t): \varphi \in \mathcal{V}\}$; $\varphi[X]$ will denote the image of X under φ. Moreover, we use some of the notation, definitions, and results from the book of Massera-Schäffer [8] and the paper of Boudourides [2].

Let us denote:

by L(J,E) - the vector space of strongly measurable functions from J into E, Bochner integrable in every finite subinterval I of J, with the topology of the convergence in the mean, on every such I;

by $B(J,\mathbb{R})$ - a Banach space, provided with the norm $\|\cdot\|_{B(\mathbb{R})}$, of real-valued measurable functions on J such that

(1) $B(J,\mathbb{R})$ is stronger than $L(J,\mathbb{R})$ (see [8], p. 35), (2) $B(J,\mathbb{R})$ contains all essentially bounded functions with compact support, and (3) if $u \in B(J,\mathbb{R})$ and v is a real-valued measurable function on J with $|v| \leq |u|$, then $v \in B(J,\mathbb{R})$ and $\|v\|_{B(\mathbb{R})} \leq \|u\|_{B(\mathbb{R})}$;

by B_0 - the Banach space of all strongly measurable functions $u: J \longrightarrow E$ such that $\|u\| \in B(J,\mathbb{R})$ provided with the norm $\|u\|_{B(E)} = \| \, \|u\| \, \|_{B(\mathbb{R})}$;

by C_0 - the Banach space of bounded continuous functions from J to E, with the usual supremum norm.

Let $B^*(J,\mathbb{R})$ be the associate space to $B(J,\mathbb{R})$ i.e., $B^*(J,\mathbb{R})$ is the Banach space of all real-valued measurable functions u on J such that

$$\|u\|_{B^*(\mathbb{R})} = \sup \{\int_J |v(s)u(s)|ds: \ v \in B(J,\mathbb{R}), \ \|v\|_{B(\mathbb{R})} \leq 1\} < \infty$$

We denote by $B^*(J,E)$ the Banach space of all strongly measurable functions $u: J \longrightarrow E$ such that $\|u\| \in B^*(J,\mathbb{R})$ provided with the norm $\|u\|_{B^*(E)} = \| \, \|u\| \, \|_{B^*(\mathbb{R})}$.

We introduce the following definitions:

<u>Definition 1</u>. The pair (B_0,C_0) is called admissible (cf. [8], p. 127), if for every $b \in B_0$ there exists at least one bounded solution of $(*)$ on J.

<u>Definition 2</u>. Given any subinterval I of J, we denote by χ_I the characteristic function of I. The space $B(J,\mathbb{R})$ is called lean (cf. [8], p. 48; [12], p. 386), if for any nonnegative function $b \in B(J,\mathbb{R})$

$$\lim_{t \to \infty} \|\chi_{[t,\infty)} b\|_{B(\mathbb{R})} = 0.$$

Our result will be proved via the fixed-point theorem given below.

Denote by $C(J,E)$ the family of all continuous functions from J to E. The set $C(J,E)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of J.

We use the following fixed-point theorem (cf. [9], Theorem 2):

Let $\mathfrak{X}$ be a nonempty closed convex subset of $C(J,E)$. Let Φ be a function which assigns to each nonempty subset X of $\mathfrak{X}$ a nonnegative real number $\Phi(X)$ with the following properties:

1° $\Phi(X_1) \leq \Phi(X_2)$ whenever $X_1 \subset X_2$;

2° $\Phi(X \cup \{y\}) = \Phi(X)$ for $y \in \mathfrak{X}$;

3° $\Phi(\overline{\text{conv}}\, X) = \Phi(X)$;

4° if $\Phi(X) = 0$ then $\overline{X}$ is compact.

Suppose that T is a continuous mapping of $\mathfrak{X}$ into itself and $\Phi(T[X]) < \Phi(X)$ for an arbitrary nonempty set $X \subset \mathfrak{X}$ such that $\Phi(X) > 0$. Under these hypotheses, T has a fixed point in $\mathfrak{X}$.

3. Result. First of all, we assume that $A \in L(J, \mathcal{L}(E))$, the pair (B_0, C_0) is admissible, and $B(J, \mathbb{R})$ is lean.

Let E_0 denote the set of all points of E which are values for $t = 0$ of bounded solutions of the differential equation $y' = A(t)y$. Suppose that E_0 is closed and has a closed complement, i.e. there exists a closed subspace E_1 of E such that E is the direct sum of E_0 and E_1.

Let P be the projection of E onto E_0, and let $U: J \to \mathcal{L}(E)$ be the solution of the equation $U' = A(t)U$ with the initial condition $U(0) = I$ (the identity mapping). For any $t \in J$ we define a function $G(t,\cdot) \in L(J, \mathcal{L}(E))$ by

$$G(t,s) = \begin{cases} U(t)PU^{-1}(s) & \text{for } 0 \leq s < t, \\ -U(t)(I-P)U^{-1}(s) & \text{for } s > t. \end{cases}$$

Let $G(t,\cdot) \in B^*(J, \mathcal{L}(E))$ and $\| G(t,\cdot)\|_{B^*(\mathcal{L}(E))} \leq K$ for any $t \in J$.

Moreover, let us put: $(Fu)(t) = f(t,u(t))$ for $u \in C(J,E)$.

<u>Theorem</u>. Suppose f is a function which satisfies the following conditions:

(1) For each $x \in E$ the mapping $t \longmapsto f(t,x)$ is measurable, and for each $t \in J$ the mapping $x \longmapsto f(t,x)$ is continuous.

(2) $\| f(t,x) \| \leq \lambda(t)$ for $(t,x) \in J \times E$, where $\lambda \in B(J, \mathbb{R})$.

(3) F is continuous as a map of any bounded subset of $C(J,E)$ into the space B_0.

Let g and h be functions of J into itself such that $g \in B(J, \mathbb{R})$ with $\sup \{ \int_J \| G(t,s)\| g(s)ds : t \in J\} \leq 1$, and h is nondecreasing with $h(0) = 0$ and $h(t) < t$ for $t > 0$. Assume in addition that for any $\varepsilon > 0$, $t > 0$ and a bounded subset X of E there exists a closed subset Q of $[0,t]$ such that
$\operatorname{mes}([0,t] \setminus Q) < \varepsilon$ and

$$\alpha(f[I \times X]) \leq \sup\{g(s): s \in I\} \cdot h(\alpha(X))$$

for each closed subset I of Q.

Then for $x_0 \in E_0$ with a sufficiently small norm there exists a bounded solution y of (+) on J such that $Py(0) = x_0$.

<u>Proof</u>. By Theorem 4.1 of [7], there exists $M > 0$ such that every bounded solution of $y' = A(t)y$ satisfies the estimate $\| y(t)\| \leq M \| y(0)\|$ for $t \in J$. Now, choose a positive number $r > K \|\lambda\|_{B(\mathbb{R})}$ and assume that $x_0 \in E_0$ with $\|x_0\| \leq$
$\leq M^{-1}(r - K \|\lambda\|_{B(\mathbb{R})})$.

Denote by $\mathfrak{X}$ the set of all $u \in C(J,E)$ such that $\|u(t)\| \leq r$ on J and

$$\|u(t_1) - u(t_2)\| \leq r \left| \int_{t_1}^{t_2} \|A(s)\| ds \right| + \left| \int_{t_1}^{t_2} \lambda(s) ds \right|$$

for t_1, t_2 in J. Define a mapping T as follows: for $u \in \mathfrak{X}$,

$$(Tu)(t) = U(t)x_0 + \int_J G(t,s)(Fu)(s)ds.$$

Let $u \in \mathfrak{X}$. For $t \in J$, by the Hölder inequality ([8], Theorem 22.M), we obtain

$$\begin{aligned}\|(Tu)(t)\| &\leq \|U(t)x_0\| + \int_J \|G(t,s)\| \ \|(Fu)(s)\| ds \leq \\ &\leq M\|U(0)x_0\| + \int_J \|G(t,s)\| \lambda(s)ds \leq \\ &\leq M\|x_0\| + K\|\lambda\|_{B(R)} \leq r.\end{aligned}$$

By Theorem 2 of [2] the function Tu is a bounded solution of the differential equation $y' = A(t)y + (Fu)(t)$. Hence

$$\|(Tu)(t_1) - (Tu)(t_2)\| \leq$$

$$\leq \left| \int_{t_1}^{t_2} \|A(s)(Tu)(s) + (Fu)(s)\| ds \right| \leq$$

$$\leq r \cdot \left| \int_{t_1}^{t_2} \|A(s)\| ds \right| + \left| \int_{t_1}^{t_2} \lambda(s) ds \right|$$

on J, and therefore $Tu \in \mathfrak{X}$.

For $u,v \in \mathfrak{X}$ and $t \in J$,

$$\|(Tu)(t) - (Tv)(t)\| \leq$$

$$\leq \int_J \|G(t,s)\| \ \|(Fu)(s) - (Fv)(s)\| ds \leq K\|Fu - Fv\|_{B(E)}.$$

From this we conclude that T is continuous as a map of $\mathfrak{X}$ into itself.

Put

$$\Phi(V) = \sup \{\alpha(V(t)) : t \in J\}$$

for a nonempty subset V of $\mathfrak{X}$. It is not hard to see that the function Φ has the properties $1^o - 4^o$ listed in Section 2. To apply our fixed-point theorem it remains to be shown that $\Phi(T[V]) < \Phi(V)$ whenever $\Phi(V) > 0$.

Assume V is a nonempty subset of $\mathfrak{X}$. Fix $t > 0$ and $\varepsilon > 0$. Since $B(J, \mathbb{R})$ is lean, $K \| \chi_{[a,\infty)} \lambda \|_{B(\mathbb{R})} < \varepsilon$ for some $a \geq t$. Let $\delta = \delta(\varepsilon) > 0$ be a number such that

$$\int_D \| G(t,s) \| \lambda(s)ds < \varepsilon$$

for each measurable $D \subset [0,a]$ with mes $(D) < \delta$. By the Luzin theorem there exists a closed subset Z_1 of $[0,a]$ with mes $([0,a] \setminus Z_1) < \delta/2$ and the function g is continuous on Z_1.

Let $X_0 = \bigcup \{V(s): 0 \leq s \leq a\}$. By our comparison condition, there exists a closed subset Z_2 of $[0,a]$ such that mes $([0,a] \setminus Z_2) < \delta/2$ and

$$\alpha(f[I \times X_0]) \leq \sup \{g(s): s \in I\} \cdot h(\alpha(X_0))$$

for each closed subset I of Z_2.

Define: $D = D_1 \cup D_2$, $Z = [0,a] \setminus D$, where $D_i = [0,a] \setminus Z_i$ $(i = 1,2)$. We have

$$\alpha(\{\int_D G(t,s)(Fu)(s)ds: u \in V\}) \leq$$
$$\leq \operatorname{diam} (\{\int_D G(t,s)(Fu)(s)ds: u \in V\}) \leq$$
$$\leq 2 \cdot \sup \{\| \int_D G(t,s)(Fu)(s)ds \| : u \in V\} \leq$$
$$\leq 2 \cdot \int_D \| G(t,s) \| \lambda(s)ds < 2\varepsilon$$

and

$$\alpha(\{\int_a^\infty G(t,s)(Fu)(s)ds: u \in V\}) \leq$$
$$\leq 2 \cdot \int_a^\infty \| G(t,s) \| \lambda(s)ds \leq 2K \| \chi_{[a,\infty)} \lambda \|_{B(\mathbb{R})} < 2\varepsilon .$$

Let

$$c_1 = \sup \{g(s): s \in Z\}, \quad c_2 = \sup \{\| G(t,s) \| : s \in Z\}.$$

Since Z is compact, for any given $\varepsilon' > 0$ there exists a $\eta > 0$

such that $|s_1' - s_1''| < \eta$ with $s_1', s_1'' \in [0,t] \cap Z$, $|s_2' - s_2''| <$ $< \eta$ with $s_2', s_2'' \in [t,a] \cap Z$ and $|s' - s''| < \eta$ with s', $s'' \in Z$ implies $c_1 \alpha(X_0) \| G(t,s_j') - G(t,s_j'') \| < \varepsilon'$ $(j = 1,2)$ and $c_2 \alpha(X_0) | g(s') - g(s'')| < \varepsilon'$.

Let $I_i = [t_{i-1}, t_i] \setminus D$ $(i = 1.2, \dots, m)$, where

$$0 = t_0 < t_1 < \dots < t_l = t < \dots < t_m = a$$

with $|t_i - t_{i-1}| < \eta$. We shall prove below that

$$\alpha(\bigcup \{G(t,s) f[I_i \times X_0] : s \in I_i\}) \leq$$

$$\leq \sup\{\| G(t,s) \| : s \in I_i\} \cdot \alpha(f[I_i \times X_0]).$$

In fact, for $\varepsilon_0 > 0$ there exist a number $\eta_0 > 0$ and sets W_j, $j = 1,2,\dots,n$, such that

$$f[I_i \times X_0] = \bigcup_{j=1}^{n} W_j, \ \operatorname{diam} W_j < \varepsilon_0 + \alpha(f[I_i \times X_0])$$

and

$$\| G(t,\sigma') - G(t,\sigma'') \| \cdot \sup\{\|x\| : x \in f[I_i \times X_0]\} < \varepsilon_0$$

for σ', $\sigma'' \in I_i$ with $|\sigma' - \sigma''| < \eta_0$. Divide the interval I_i into r parts $d_1 < d_2 < \dots < d_{r+1}$ in such a way that $|d_{k+1} - d_k| < \eta_0$ $(k = 1,2,\dots,r)$. Furthermore, let us denote by X_{jk} $(j = 1,2,\dots,n;\ k = 1,2,\dots,r)$ the set of all $x \in E$ such that there exists a point $w \in W_j$ with $\| x - G(t,d_k) w \| < \varepsilon_0$.

Let $\xi = G(t,s_0) z_0$, where $s_0 \in [d_q, d_{q+1}]$ and $z_0 \in W_p$. Then

$$\| \xi - G(t,d_p) z_0 \| \leq \| G(t,s_0) - G(t,d_p) \| \, \| z_0 \| < \varepsilon_0$$

hence $\xi \in X_{pq}$. Consequently,

$$\bigcup \{G(t,s) f[I_i \times X_0] : s \in I_i\} \subset \bigcup_{j=1}^{n} \bigcup_{k=1}^{r} X_{jk}.$$

If $\| x_\rho - G(t,d_k) w_\rho \| < \varepsilon_0$ $(\rho = 1,2)$ with $x_\rho \in X_{jk}$ and $w_\rho \in W_j$, then

$$\| x_1 - x_2 \| \leq \| x_1 - G(t,d_k) w_1 \| + \| G(t,d_k) w_1 - G(t,d_k) w_2 \| +$$

$+ \|G(t,d_k)w_2 - x_2\| <$

$< 2\varepsilon_0 + \sup\{\|G(t,s)\| : s \in I_i\} \cdot \operatorname{diam}(W_j) <$

$< 2\varepsilon_0 + [\varepsilon_0 + \alpha(f[I_i \times X_0])] \cdot \sup\{\|G(t,s)\| : s \in I_i\}.$

Therefore,

$$\alpha(\bigcup\{G(t,s)f[I_i \times X_0] : s \in I_i\}) \leq 2\varepsilon_0 + [\varepsilon_0 +$$

$$+ \alpha(f[I_i \times X_0])] \cdot \sup\{\|G(t,s)\| : s \in I_i\}$$

and our claim is proved.

Applying the integral mean value theorem, we get

$$\alpha(\{\int_Z G(t,s)(Fu)(s)ds : u \in V\}) \leq$$

$$\leq \alpha(\sum_{i=1}^{m} \operatorname{mes}(I_i)\, \overline{\operatorname{conv}}\, (\bigcup\{G(t,s)f[I_i \times X_0] : s \in I_i\})) \leq$$

$$\leq \sum_{i=1}^{m} \operatorname{mes}(I_i)\alpha(\bigcup\{G(t,s)f[I_i \times X_0] : s \in I_i\}) \leq$$

$$\leq \sum_{i=1}^{m} \operatorname{mes}(I_i)\|G(t,\sigma_i)\|\, g(\tau_i)h(\alpha(X_0)),$$

where σ_i, τ_i are points in I_i such that

$$\|G(t,\sigma_i)\| = \sup\{\|G(t,s)\| : s \in I_i\} \text{ and}$$

$$g(\tau_i) = \sup\{g(s) : s \in I_i\}.$$

Now, from the above, we obtain

$$\alpha(T[V](t)) \leq \alpha(\{\int_D G(t,s)(Fu)(s)ds : u \in V\}) +$$

$$+ \alpha(\{\int_Z G(t,s)(Fu)(s)ds : u \in V\}) +$$

$$+ \alpha(\{\int_a^\infty G(t,s)(Fu)(s)ds : u \in V\}) <$$

$$< 4\varepsilon + h(\alpha(X_0)) \sum_{i=1}^{m} \int_{I_i} (\|G(t,s)\|\, g(s) +$$

$$+ c_1\|G(t,s) - G(t,\sigma_i)\| + c_2|g(s) - g(\tau_i)|)ds.$$

Suppose $\alpha(X_0) > 0$. From the above, it follows that

$$\alpha(T[V](t)) <$$

$$< 4\varepsilon + \sum_{i=1}^{m} \int_{I_i} (\|G(t,s)\|\, g(s) + \frac{2\varepsilon'}{\alpha(X_0)}) \cdot h(\alpha(X_0))ds <$$

$< 4\varepsilon + h(\alpha(X_0)) \int_Z \| G(t,s) \| g(s)ds + 2\varepsilon' \cdot \text{mes}\,(Z).$

Since V is almost equicontinuous and bounded, we can apply Lemma 2.2 of Ambrosetti [1] to get

$$\alpha(X_0) = \sup \{\alpha(V(s)) :\ 0 \le s \le a\} \le \Phi(V).$$

Therefore

$$\alpha(T[V](t)) < 4\varepsilon + h(\Phi(V)) \int_Z \| G(t,s) \| g(s)ds + 2\varepsilon' \cdot \text{mes}\,(Z),$$

and we obtain $\alpha(T[V](t)) \le h(\Phi(V))$. If $\alpha(X_0) = 0$, then $\alpha(T[V](t)) \le 0 = h(0) \le h(\Phi(V))$. This proves

$$\alpha(T[V](t)) \le h(\Phi(V)) \text{ for each } t \in J;$$

hence $\Phi(T[V]) \le h(\Phi(V))$.

The set $\mathfrak{X}$ is a closed and convex subset of C(J,E). Thus all assumptions of our fixed-point theorem are satisfied; T has a fixed point in $\mathfrak{X}$ which ends the proof.

Remark. Our result may be applied to the important case, when B_0 is any Orlicz space L_φ generated by a convex φ-function such that $\lim_{u \to 0} \varphi(u)/u = 0$ and $\lim_{u \to \infty} \varphi(u)/u = \infty$.

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Institute of Mathematics A. Mickiewicz University, Matejki 48/49, 60-769 Poznan, Poland

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