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THE LATTICE OF R-SUBALGEBRAS OF A BOUNDED DISTRIBUTIVE LATTICE
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Abstract: Using Priestley's duality, we investigate the lattice $S_R(L)$ of the $\{0,1\}$-sublattices of a given bounded distributive lattice L which are closed under relative comp-plementation. We characterize those bounded distributive lattices L such that $S_R(L)$ is semimodular, modular, distributive or Boolean.

Key words: Distributive lattice - Relative complementation - Priestley's duality - Congruences on partially ordered spaces.

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Introduction. In his study on Boolean lattices R-genera-ted by distributive lattices, Grätzer considers particular $\{0,1\}$-sublattices of a given bounded distributive lattice, na-mely those which are closed under relative complementation. The purpose of this paper is to study these sublattices, which we call R-subalgebras. It turns out that Priestley's duality is a well-adapted tool to achieve this aim.

In Section 1, we introduce the concept of congruence on a Priestley space, which is dual to that of R-subalgebra; the lattice of all R-subalgebras of a bounded distributive lattice is dually isomorphic to the lattice Con(X) of all congruences on the dual X of L.

Section 2 is devoted to the study of Con(X). In particular we characterize those Priestley spaces whose congruence lattice is semi-modular, modular or distributive respectively. We translate these results in terms of R-subalgebras in Section 3.

We adopt standard set theoretic notations. Let us however recall some of them. For a set X, we denote by $|X|$ its cardinal and by Eq(X) its equivalence lattice. If $\Theta \in Eq(x)$, $x \in X$ and $E \subseteq X$, we write x^Θ for the Θ-class of x and $E^\Theta = \cup\{x^\Theta \mid x \in E\}$; E is Θ-saturated if $E^\Theta = E$. If $X = (X, \leq)$ is a poset, $p \prec q$ means that q covers p and $p \parallel q$ means that p and q are not comparable. We say that $E \subseteq X$ is convex if $x \leq z \leq y$ and $x, y \in E$ imply that $z \in E$. An order connected component (o.c.c.) of X is a subset E of X which is minimal with respect to the property of being both increasing and decreasing. Finally, the n-element chain is denoted by $\underline{n}$.

1. A Duality for R-subalgebras of a bounded distributive lattice

1.1. Definition. Let $\mathbb{D}$ denote the category of bounded distributive lattices and $\{0,1\}$-homomorphisms. If $L \in \mathbb{D}$ and A is a $\{0,1\}$-sublattice of L, then A is said to be an R-subalgebra of L if it is closed under relative complementation (when the latter is defined). Other ways of defining R-subalgebras are given in [4]. The set of all R-subalgebras of a lattice L in $\mathbb{D}$, ordered by inclusion, is an algebraic lattice, the study of which is the purpose of the present paper. We denote it by $\mathcal{S}_R(L)$.

In [6], H.A. Priestley establishes a duality between $\mathbb{D}$ and the category $\mathbb{P}$ of Priestley (i.e. compact totally order disconnected) spaces and order-preserving continuous maps. The functors

- 2 -

$\mathcal{P}: \mathbb{D} \rightarrow \mathbb{P}$ and $\mathcal{O}: \mathbb{P} \rightarrow \mathbb{D}$ which realize the duality are described as follows: if $L \in \mathbb{D}$, $\mathcal{P}(L)$ is the ordered set of all prime ideals of L, suitably topologized , whereas, for $X \in \mathbb{P}$, $\mathcal{O}(X)$ is the lattice of all clopen order-ideals of X. If f is a morphism in $\mathbb{D}$ (resp. $\mathbb{P}$), its dual map is defined by $\mathcal{P}(f) = = f^{-1}$ (resp. $\mathcal{O}(f) = f^{-1}$). We refer to [7] for the fundamental facts on Priestley's duality but we usually follow the notations of [2].

1.2. <u>Definition</u>. Let $X \in \mathbb{P}$. Suppose (X', τ') is a topological space and π is an onto continuous map $X \longrightarrow X'$. An order $\leqslant'$ on X' is said to be <u>compatible</u> with τ' and π if $X' = = (X', \tau', \leqslant')$ is in $\mathbb{P}$ and if π is order-preserving (hence a morphism in $\mathbb{P}$).

1.3. <u>Lemma</u>. <u>Let</u> $L \in \mathbb{D}$ <u>and</u> $A \in \mathcal{P}_R(L)$. <u>Denote by</u> $X = = (X, \tau , \leqslant)$ <u>the dual of</u> A <u>and by</u> π <u>the dual map of the inclusi-on map</u> $A \longrightarrow L$. <u>Then</u> $\leqslant$ <u>is the least order on</u> X <u>which is compatible with</u> τ <u>and</u> π .

Proof. Suppose $\leqslant'$ is an order on X which is compatible with τ and π . Denote by A' the dual lattice of $(X, \tau, \leqslant')$, considered as a $\{0,1\}$-sublattice of L. By [7], both A and A' R-generate the Boolean algebra whose dual is (X, τ). Since A is closed under relative complementation, it follows from [3], p. 89, that A contains A'. In other words, $\mathcal{O}(X, \tau , \leqslant') \supseteq \supseteq \mathcal{O}((X, \tau, \leqslant')$, which implies that $\leqslant$ is contained in $\leqslant'$.

By 1.3, the dual of an R-subalgebra A of a lattice $L \in \mathbb{D}$ is determined by the canonical epimorphism $\pi : \mathcal{P}(L) \longrightarrow \mathcal{P}(A)$. Therefore, the general concept of separating set [1] may be advantageously replaced by the simpler one of congruence.

1.4. Definition. Let $X \in \mathbb{P}$ and $\Theta \in \mathrm{Eq}(X)$. Then Θ deserves the name of congruence if there exists a topology τ' and an order $\leq'$ on X such that

i) the natural map $\pi : X \rightarrow X/\Theta$ is continuous, and

ii) $\leq'$ is compatible with τ' and π.

We denote by $\mathrm{Con}(X)$ the set of all congruences on X. Obviously, ω (the identity relation) and ι (the universal relation) are always congruences. Since the intersection of any subfamily of $\mathrm{Con}(X)$ is again a congruence, $\mathrm{Con}(X)$ is a complete lattice, but it need not be a sublattice of $\mathrm{Eq}(X)$. It is also worth to note that, if $\Theta \in \mathrm{Con}(X)$, the topology τ' of the definition is necessarily the quotient topology, which we shall denote by τ_Θ. Moreover, among all orders $\leq'$ compatible with τ' and π, there always exists a least one, that we shall denote by $\leq_\Theta$ (it suffices to consider the R-subalgebra of $\mathcal{O}(X)$ generated by $\mathcal{O}((X/\Theta, \tau', \leq'))$ and to apply 1.3). We shall now describe $\leq_\Theta$.

1.5. Notation. Let $X \in \mathbb{P}$ and $\Theta \in \mathrm{Eq}(X)$. We denote by $\mathcal{O}(X,\Theta)$ the set of all clopen order-ideals of X which are Θ-saturated and we define on X/Θ a quasi-order $\leq_\Theta$ as follows: $x^\Theta \leq_\Theta y^\Theta$ if, for all $U \in \mathcal{O}(X,\Theta)$, $U \ni y$ implies $U \ni x$.

1.6. Lemma. Let $X \in \mathbb{P}$ and $\Theta \in \mathrm{Eq}(X)$. The following assertions are equivalent:

(i) $\Theta \in \mathrm{Con}(X)$;

(ii) $\leq_\Theta$ is antisymmetric;

(iii) $\leq_\Theta$ is the least order compatible with τ_Θ and π;

(iv) if $x \Theta y$ fails, then x and y can be separated by some member of $\mathcal{O}(X,\Theta)$.

Proof. (i) $\Rightarrow$ (ii). Let $\leq'$ be an order on X/Θ which is compatible with τ_Θ and π. If $x^\Theta \nleq' y^\Theta$, there exists $V \in \mathcal{O}(X/\Theta, \tau_\Theta, \leq'))$ with $V \ni y^\Theta$ and $V \not\ni x^\Theta$. If $U = \pi^{-1}(V)$, then

$U \in \mathcal{O}(X, \Theta)$, $U \ni y$ and $U \not\ni x$, which shows that $x^{\Theta} \not\leq_{\Theta} y^{\Theta}$. Consequently, $\leq_{\Theta}$ is contained in $\leq'$ and therefore is antisymmetric.

(ii) $\rightarrow$ (iii). It is clear that $\leq_{\Theta}$ is an order on X/Θ which is compatible with τ_{Θ} and π. The proof of (i) $\Rightarrow$ (ii) shows that it is the least one.

Finally, (iii) $\rightarrow$ (i) and (ii) $\Leftarrow$ (iv) are trivial.

1.7. <u>Theorem</u>. <u>If $L \in \mathbb{D}$ and if X is its dual space, then there exists a canonical dual isomorphism $\mathcal{S}_R(L) \rightarrow \mathrm{Con}(X)$. In particular, $\mathrm{Con}(X)$ is dually algebraic.</u>

Proof. We may assume that $L = \mathcal{O}(X)$. Let us define $h:\mathrm{Con}(X) \rightarrow \mathcal{S}_R(L)$ by $h(\Theta) = \mathcal{O}(X, \Theta)$. Clearly, h is one-to-one and order-preserving (see 1.6(iv)).

Let now $A \in \mathcal{S}_R(L)$. Define Θ to be the kernel of $\mathcal{P}(\mathrm{id})$, where id is the inclusion map $A \rightarrow L$. In other words, $x \Theta y$ if and only if $U \ni x \Leftarrow U \ni y$ for all $U \in A$. We wish to show that $h(\Theta) = A$. It is clear that $A \subseteq h(\Theta)$. Let $U \in h(\Theta)$. For each $x \in U$ and $y \notin U$, there exists by 1.6(iv) either $U_{xy} \in A$ such that $U_{xy} \ni x$ and $U_{xy} \not\ni y$, or $V_{xy} \in A$ such that $V_{xy} \ni y$ and $V_{xy} \not\ni x$. If y is fixed, the sets U_{xy} and $-V_{xy}(x \in U)$ form an open covering of U which, by compactness, has a finite subcover. This gives rise to elements $U_y \in A$, $V_y \in A$ such that $U \subseteq U_y \cup -V_y \subseteq -\{y\}$. Hence, $-U = \cup \{V_y \cap -U_y | y \notin U\}$. Again by compactness, it follows that U is the intersection of finitely many $U_y \cup -V_y$. Therefore, U is in the Boolean algebra R-generated by A. Since $U \in \mathcal{O}(X)$, $U \in A$ by [3].

The map $A \mapsto \Theta$ is obviously order-preserving and the proof is over.

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2. The congruence lattice of a Priestley space

2.1. **Notation.** Let $X \in \mathbb{P}$. If $E \subseteq X$, we denote by $\Theta(E)$ the equivalence on X generated by $E \times E$ and by $\phi(E)$ the equivalence $\Theta(E) \cup \Theta(-E)$. If $E = \{p,q\}$ we write $\Theta(p,q)$ instead of $\Theta(\{p,q\})$.

2.2. **Lemma.** If $X \in \mathbb{P}$ and $E \subseteq X$, then $\Theta(E) \in \mathrm{Con}(X)$ if and only if E is closed and convex.

Proof. Any congruence class is closed and convex. Hence the condition is necessary. Suppose now that $(x,y) \notin \Theta(E)$ where E is closed and convex. To find $U \in \Theta(X,\Theta)$ separating x and y, it suffices to distinguish the possible positions of x and y relatively to E.

2.3. **Theorem.** If $X \in \mathbb{P}$, then $\mathrm{Con}(X)$ is atomistic. Its atoms are the equivalences $\Theta(p,q)$ where $p \parallel q$ or $p \prec q$.

Proof. Let us first show that any closed and convex subset E of X which is not reduced to a singleton contains a pair $\{p,q\}$ where $p \parallel q$ or $p \prec q$. This is clear if E is not a chain. If E is a chain, it is a Boolean chain, in which jumps $p \prec q$ exist in abundance ([5]).

To prove atomisticity, note that one has $\Theta = \bigvee \{\Theta(E) \mid E$ is a Θ-class$\}$ for each $\Theta \in \mathrm{Con}(X)$. Hence it remains to prove that, if E is closed and convex, $\Theta(E) = \bigvee \{\Theta(p,q) \in \mathrm{Con}(X) \mid p,q \in E\}$. Let $\Phi \in \mathrm{Con}(X)$ be such that $\Phi \geq \Theta(p,q)$ for all $p,q \in E$ with $\Theta(p,q) \in \mathrm{Con}(X)$. If $\Phi \neq \Theta(E)$, there exist x, y in E for which $x \, \Phi \, y$ fails. Consequently, x and y can be separated by some $U \in \Theta(X,\Phi)$. If p is maximal in $U \cap E$ and q minimal in $-U \cap E$, then $p,q \in E$ and $\Theta(p,q) \in \mathrm{Con}(X)$ whence $\Theta(p,q) \leq \Phi$, a contradiction.

2.4. **Theorem**. If $X \in \mathbb{P}$, then Con(X) is dually atomistic. Its dual atoms are the equivalences $\bar{\Phi}(U)$, where $U \in \mathcal{O}(X) - \{\emptyset, X\}$.

Proof. It is clear that $\bar{\Phi}(U) \in$ Con(X) if and only if $U \in \mathcal{O}(X)$ (use 1.6(iv). It suffices now to show that, if $\Theta \in$ Con(X), then $\Theta = \bigwedge \{ \bar{\Phi}(U) \mid U \in \mathcal{O}(X, \Theta) \}$. If $U \in \mathcal{O}(X, \Theta)$, then $\Theta \leqslant \bar{\Phi}(U)$. Conversely, if $\bar{\Phi} \leqslant \bar{\Phi}(U)$ for each $U \in \mathcal{O}(X, \Theta)$, and if $x \bar{\Phi} y$, then $x \Theta y$ by 1.6(iv) . Hence $\bar{\Phi} \leqslant \Theta$, which completes the proof.

The following result shows that the semimodularity of Con(X) depends only on the order on X and not on its topology. (A lattice L is called **semimodular** if and only if it satisfies the following condition for all $a, b \in L: a \wedge b \prec a \implies b \prec a \vee b$.)

2.5. **Theorem**. If $X \in \mathbb{P}$, then Con(X) is semimodular if and only if either

(i) X is order-isomorphic to an ordinal sum $A \oplus C \oplus A'$, where A and A' are (possibly empty) antichains and C is a bounded chain, or

ii) X = Min$X \cup$ Max$X$ and either $|X - \text{Min} X| \leqslant 1$ or $|X - \text{Max} X| \leqslant 1$.

Proof. Suppose first that Con(X) is semimodular. We proceed in four steps.

a) There cannot exist in X elements x, y, z, t with $x < y$, $z < t$, $x \parallel t$ and $y \parallel z$ (otherwise $\Theta(x, t) > \Theta(y, z) \wedge \Theta(x, t) = \omega$ and $\Theta(y, z) \vee \Theta(x, t) > \Theta([x, y] \cup \{z\}) > \Theta(y, z)$). In particular, there exists at most one o.c.c. which is not reduced to a singleton. Let us denote it by X_0.

b) If $p, q \in X_0$ and $p \parallel q$, then for each $x \in X_0$, $x > p$ (resp. $x < p$) implies $x > q$ (resp. $x < q$). Suppose on the contrary that,

for some $r \in X_0$, one has $r > p$ and $r \not> q$ (which implies $r \parallel q$). We distinguish three possibilities.

If $[p) \wedge [q) \neq \emptyset$, it contains some minimal element t. Necessarily, either $r \parallel t$ or $r < t$. In the first case, we have $\Theta([p,t]) \vee \Theta(q,r) > \Theta([p,t] \cup [q,t]) > \Theta([p,t])$. In the second case, we have $\Theta([q,t]) \vee \Theta(p,q) > \Theta([q,t] \cup [r,t]) > \Theta([q,t])$. Both inequalities contradict the fact that $\mathrm{Con}(X)$ is semimodular.

If $[p) \cap [q) = \emptyset$ and $(r] \cap (q] \neq \emptyset$, choose some maximal element t in $(r] \cap (q]$. If $t < p$, then $\Theta([t,q]) \vee \Theta(q,r) > \Theta([t,q] \cup [t,p]) > \Theta([t,q])$. If $t \parallel p$, then $\Theta([t,r]) \vee \Theta(p,q) > \Theta([t,r] \cup [p,r]) > \Theta([t,r])$. Here again this is not possible because of the semimodularity of $\mathrm{Con}(X)$.

It remains to consider the case where $[p) \cap [q) = \emptyset$ and $(r] \cap (q] = \emptyset$. Since $q \in X_0$, there exists in $\mathrm{Min}X \cup \mathrm{Max}X$ some element $t \neq q$ which is comparable with q, say $q < t$. The existence of the elements p, r, q, t contradicts a).

c) Suppose now $X \neq \mathrm{Min}X \cup \mathrm{Max}X$. The only o.c.c. of X are $\emptyset$ and X itself. Otherwise choose $x < y < z$ and some t not belonging to the same o.c.c. as x. Then $\Theta(x,t) \vee \Theta(z,t) > \Theta([x,y)] \cup \{t\}) > \Theta(x,t)$, which is not possible.

Moreover, $C = X-(\mathrm{Min}X \cup \mathrm{Max}X)$ is a chain. Indeed if p, q are non comparable elements of C, let t (resp. u) be minimal (resp. maximal) in $[p) \cap [q)$ (resp. $(p] \cap (q]$) (these sets are not empty by b)). Then $\Theta(p,t) \vee \Theta(p,u) > \Theta(\{p,q,t\}) > \Theta(p,t)$ and this again is not possible.

As a consequence, X is of the type i) as required.

d) If $X = \mathrm{Min}X \cup \mathrm{Max}X$, we have to prove that $|X-\mathrm{Min}X| \leqslant 1$ or $|X-\mathrm{Max}X| \leqslant 1$. Suppose on the contrary that there exist distinct elements x, y in $\mathrm{Min}X-\mathrm{Max}X$ and z, t in $\mathrm{Max}X-\mathrm{Min}X$. By b), we may assume that $x < z$, $x < t$, $y < z$ and $y < t$. Then $\Theta(x,z) \vee \Theta(y,t) >$

$> \Theta\{(x,yz)\} > \Theta(x,z)$ which is absurd.

Assume now that X satisfies either i) or ii). We have to prove that if $\Phi, \Theta \in \text{Con}(X)$, then $\Phi \wedge \Theta \prec \Theta$ implies $\Phi \wedge \Theta \prec \Theta$. It is not difficult to show that the third isomorphism theorem holds in $\mathbb{P}$ and we may assume $\Phi \wedge \Theta = \omega$. We shall prove the following stronger result: if $\Phi \in \text{Con}(X)$ and if Θ is an atom in Con(X), then the supremum $\Phi \wedge_q \Theta$ of Φ and Θ in Eq(X) is a congruence. To achieve this result, let us suppose $\Theta = \Theta(p,q)$ where $p \parallel q$ or $p \prec q$. We first show that it is not possible to have $(*)$ $p^\Phi < y^\Phi < q^\Phi$ for some $y \in X$ (here, $<$ is written instead of $<_\Phi$). The proof is carried on ab absurdo.

a) Suppose first that X satisfies i). If $p \parallel q$, then $\{p,q\} \subseteq \text{MinX}$ or $\{p,q\} \subseteq \text{MaxX}$, say $\{p,q\} \subseteq \text{MinX}$. It results from $(*)$ that $p^\Phi \cup y^\Phi \subseteq \text{MinX}$. Let t be the least element of X-MinX. Then $y^\Phi < t^\Phi$, and there exists $V \in \mathcal{O}(X, \Phi)$ such that $V \ni y$ and $V \not\ni t$. Moreover, since $p^\Phi < y^\Phi$, there exists $W \in \mathcal{O}(X, \Phi)$ such that $W \ni p$ and $W \not\ni y$. If $U = V-W$, then $U \in \mathcal{O}(X, \Phi)$, $U \ni y$ and $U \not\ni p$, which contradicts $p^\Phi < y^\Phi$.

If $p \prec q$ and $p \in \text{MinX}$, then $q = t$ and we have seen that $p^\Phi < y^\Phi < t^\Phi$ is not possible. Hence we may assume that $p \in X - (\text{MinX} \cup \text{MaxX})$. In the same way, we may assume that $q \in X - (\text{MinX} \cup \text{MaxX})$. Since $X-(\text{MinX} \cup \text{MaxX})$ is a chain, y is comparable with p and q and $(*)$ implies $p < y < q$, which contradicts $p \prec q$.

b) Suppose now that X satisfies (ii). Obviously, $(*)$ prevents X from being an antichain. By (ii), we may assume that $X-\text{MaxX} = \{m\}$ for some m. Let us show that $x^\Phi < y^\Phi$ implies x m (and this contradicts $(*)$). If not, then either $x^\Phi < m^\Phi$ or $x \not< m$. The first possibility cannot occur because, if $U \in \mathcal{O}(X)$ and $U \not\ni m$, then $-U \in \mathcal{O}(X)$. Hence there exists $V \in \mathcal{O}(X, \Phi)$ such

- 9 -

that $V \ni m$ and $V \not\ni x$. Since $x^\Phi < y^\Phi$, there also exists $W \in \mathcal{O}(X, \Phi)$ such that $W \ni x$ and $W \not\ni y$. If $U = V \cup -W$, then $U \in \mathcal{O}(X, \Phi)$, $U \ni y$ and $U \not\ni x$, which contradicts $x^\Phi < y^\Phi$.

We are now in a position to prove that $\Phi \vee_{eq} \Theta(p,q) \in$ $\in \mathrm{Con}(X)$. Let $\alpha = \Phi \vee_{eq} \Theta(p,q)$ and suppose that $x \propto y$ fails. To separate x and y by some member of $\mathcal{O}(X, \alpha)$ we have to consider the various positions of x and y relative to p and q. As an example, let us assume $x^\Phi = p^\Phi$, $y^\Phi \neq p^\Phi$ and $y^\Phi \neq q^\Phi$.

If $p^\Phi \nleq y^\Phi$ and $q^\Phi \nleq y$, there exists $U \in \mathcal{O}(X, \Phi)$ such that $U \ni y$, $U \not\ni x$ and $\{p,q\} \subseteq -U$, which implies $U \in \mathcal{O}(X, \alpha)$. If $p^\Phi < y^\Phi$ (same argument if $q^\Phi < y^\Phi$), then $y^\Phi \nleq p^\Phi$ and $y \nleq q^\Phi$ (otherwise $p^\Phi < y^\Phi < q^\Phi$) and we may argue as above.

Theorem 2.5 enables us to characterize those $X \in \mathbb{P}$ for which $\mathrm{Con}(X)$ is geometric (i.e. $\mathrm{Con}(X)$ is semimodular, complete, atomistic and all atoms of $\mathrm{Con}(X)$ are compact).

2.6. <u>Theorem</u>. Let $X \in \mathbb{P}$. <u>Then</u> $\mathrm{Con}(X)$ <u>is geometric if and only if it has one of the forms</u> (i) <u>or</u> (ii) <u>of 2.5 and moreover,</u> $\mathrm{Min} X \cup \mathrm{Max} X$ <u>is finite</u>.

Proof. Suppose $\mathrm{Con}(X)$ is geometric. By 2.5, it remains to prove that $\mathrm{Min} X \cup \mathrm{Max} X$ is finite. Assume on the contrary that $\mathrm{Min} X$ is infinite. If $\mathrm{Min} X$ is not closed, let $p \in \mathrm{Min} X$ and let q be the least element of $X - \mathrm{Min} X$. Then $\Theta(p,q)$ is not compact since $\Theta(p,q) \leq \vee T$, where $T = \{\Theta(x,y) \mid x,y \in \mathrm{Min} X\}$ whereas $\Theta(p,q) \nleq T'$ for any finite subset T' of T.

If $\mathrm{Min} X$ is closed and thus compact, there exists $p \in \mathrm{Min} X$ such that $\{p\}$ is not open. Let q be an element of $\mathrm{Min} X - \{p\}$. If $T = \{\Theta(x,y) \mid x,y \in \mathrm{Min} X - \{p\}\}$, we conclude as above.

Conversely, if X satisfies i) (case ii) is trivial) of 2.5 and $\mathrm{Min}\, X \cup \mathrm{Max}\, X$ is finite, then each atom is compact. To show

this, let T be a set of atoms in Con(X) such that $\theta(p,q) \not\leq$ T. We consider two possibilities.

a) If $\{p,q\} \subsetneq (t]$ where t is the least element of X-MinX, then $\theta(p,q) \leq \vee\{\theta(x,y) \in T \mid \{x,y\} \subseteq (t]\}$.

b) If $\{p,q\} \subseteq$ C where C is the Boolean chain described in 2.5 i), then necessarily $\theta(p,q) \in$ T because $(p] \in \mathcal{O}(X,\theta)$ for any $\theta \in T - \{\theta(p,q)\}$.

We now study the modularity of Con(X). We first need to observe that $\theta \in$ Con(X) is dually compact if and only if X/θ is finite (use Priestley's duality).

2.7. __Theorem__. __Let__ $X \in P$. __If__ X __is not the ordinal sum of__ __two 2-element antichains, then the following assertions are e-__ __quivalent__:

(i) Con(X) __is modular__;

(ii) Con(X) __is dually semimodular__;

(iii) Con(X) __is dually geometric__;

(iv) __if__ Φ __and__ ψ __are dual atoms of__ Con(X), then $\Phi\wedge\psi \prec \Phi$ (__and__ $\Phi\wedge\psi \prec \psi$);

(v) __either__ $|X| \leq 3$ __or__ X __is isomorphic to a subspace of__ $A \oplus C \oplus A'$ __where__ A __and__ A' __are two-element antichains and__ C __is a bounded__ __chain__.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. Let us prove (iv) $\Rightarrow$ (v). We proceed in three steps, assuming that (iv) holds.

a) As an ordered set, C = X-(MinX$\cup$MaxX) is a chain. If not, let $x,y \in$ C be such that $x \parallel y$. Choose $x_0 \in$ MinX$\cap(x]$ and $y_0 \in$ MaxX$\cap$ $\cap[y)$. There exists $U,V \in \mathcal{O}(X)$ such that $V \supset \{x_0,y\}$, $-V \supseteq \{x,y_0\}$, $U \supset \{x_0,y\}$, $-V \supseteq \{x,x_0\}$, $U \supseteq \{x,x_0\}$ and $-U \supseteq \{y,y_0\}$. Hence

$\Phi(V) \wedge \Phi(U) < (\Phi(V) \wedge \Phi(U)) \vee \Theta(V) < \Phi(V)$, which is absurd by (iv).

b) If $|X| > 3$, then $|\mathrm{Min}X| \leq 2$ (and in the same way, $|\mathrm{Max}X| \leq 2$. Otherwise, let $x, y, z \in \mathrm{Min}X$ and $t \in X - \{x, y, z\}$. There exists $U, V \in \mathcal{O}(X)$ such that $U \supseteq \{x, y\}$ and $-U \supseteq \{z, t\}$, $V \supseteq \{x, z\}$ and $-V \supseteq \{y, t\}$. A contradiction arises as in a).

c) If $|X| > 3$ and $p \parallel q$, then for each $x \in X, x > p$ (resp. $x < p$) implies $x > q$ (resp. $x < q$). Taking into account that any element of X dominates a minimal element and is dominated by a maximal one, we may assume by a) that $\{p, q\} \subseteq \mathrm{Min}X$ or $\{p, q\} \subseteq \mathrm{Max}X$, say $\{p, q\} \subseteq \mathrm{Min}X$. Suppose that $x > p$ and $x \not> q$ (which implies $x \parallel q$). Choose $y \in X - \{p, q, x\}$. If $y \not< x$, there exist $U, V \in \mathcal{O}(X)$ such that $U \supseteq \{p, x\}, -U \supseteq \{q, y\}$, $V \supseteq \{p, q\}$ and $-V \supseteq \{x, y\}$ and we conclude as in a). The same argument holds if $x \not< y$ (interchanging x and y).

Let us now prove that (v) $\Rightarrow$ (i). If $|X| \leq 3$, then clearly $\mathrm{Con}(X)$ is modular and we may assume that $X = A \oplus C \oplus A'$ where A and A' are two-element antichains and C is a bounded chain (with least element c and greatest element d). The congruence lattice of C, is dually isomorphic to the lattice of all $\{0, 1\}$-sublattices of $\mathcal{O}(C)$, hence it is Boolean. The congruence lattice of $(c]$ (and similarly that of $[d)$) is isomorphic with M_5, the five-elements modular non distributive lattice. It remains to observe that the map $\theta \mapsto (\theta|_{(c]}, \theta|_C, \theta|_{[d)})$ is an isomorphism from $\mathrm{Con}(X)$ onto $\mathrm{Con}((c] \times \mathrm{Con}(C) \times \mathrm{Con}([d))$. This is a routine exercise.

Remark. If X is the ordinal sum of two 2-element antichains, it is easy to see that (ii), (iii) and (iv) hold but $\mathrm{Con}(X)$ is not modular. Indeed, let x_0, y_0 (resp. x_1, y_1) be the minimal (resp. maximal) elements of X. We have

$$\omega = \Theta(x_0, y_1) \wedge \Theta(y_0, y_1) \prec \Theta(x_0, x_1)$$

while

$$\theta(y_1,y_0) < \theta(\{x_0,y_0,y_1\}) < \theta(x_0,y_1) \vee \theta(y_0,y_1) = 1.$$

2.8. **Theorem**. Let $X \in \mathbf{P}$. **The following assertions are equivalent:**

i) Con(X) **is Boolean;**

ii) Con(X) **is distributive;**

iii) Con(X) **is uniquely complemented;**

iv) **either** $|X| \nleq 2$ **or** X **is a Boolean chain.**

Proof. It is clear that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) and it has been said in the proof of 2.7 that (iv) $\Rightarrow$ (i).

Let us prove (ii) $\Rightarrow$ (iv). By 2.7, either $|X| \leq 3$ or X is iso-morphic to a subspace of $A \oplus C \oplus A'$ where A,A' are two-element antichains and C is a bounded chain. It is not difficult to check that, if $|X| = 3$, then X must be a three-element chain. We may therefore suppose that $|X| > 3$. In this case, X has a least ele-ment (and for a similar reason a greatest one).
Indeed, suppose that $p,q \in \text{MinX}$. Let $r \in X-\{p,q\}$. There exist $U,V \in O(X)$ such that $U \ni q$, $-U \supseteq \{p,r\}$, $V \ni p$ and $-V \supseteq \{q,r\}$. Then $(\Phi(V) \wedge \theta(p,q)) \vee (\Phi(U) \wedge \theta(p,q)) = \omega$ and $(\Phi(V) \vee \Phi(U)) \wedge \theta(p,q) = \theta(p,q)$, which is impossible since Con(X) is distributive.

We now prove that (iii) $\Rightarrow$ (iv). First observe that, if $U \in O(X) - \{\emptyset\}$ and $a \in \text{MaxX}-U$, then $\theta(-U \cup \{a\})$ is a complement of $\theta(U)$. By (iii), any $U \in O(X) - \{\emptyset,X\}$ has a greatest element and, for dual reasons, -U has a least element. Now let x, y be non-comparable elements of X. There exist $U,V \in O(X)$ such that $x \in U-V$ and $y \in U-V$. We claim that $\{U,V\}$ is a partition of X. If not, then for instance $U \cup V \neq X$ and $U \cup V$ has a greatest element,

- 13 -

which implies $U \subseteq V$ or $V \subseteq U$ and this is impossible. Let p be
the least element of U and q the least element of V. To end
the proof, we shall show that $X = \{p,q\}$. If not, let
$r \in X - \{p,q\}$ and suppose for instance that $p \not\leq r$. There exist
$U', V' \in \mathcal{O}(X)$ such that $U' \supseteq \{p,q\}$, $U' \not\ni r$, $V' \supseteq \{q,r\}$ and
$V' \ni p$. Then $-(U' \cap V')$ has a least element and this implies
$U' \subseteq V'$ or $V' \subseteq U'$, a contradiction.

3. The lattice of R-subalgebras of a bounded distributive lattice

In this section, we dualize the results of the previous sec-
tion to obtain results on $\mathcal{S}_R(L)$, for $L \in \mathbb{D}$. We omit the
proofs which are straightforward.

3.1. **Theorem.** If $L \in \mathbb{D}$, $\mathcal{S}_R(L)$ is algebraic, atomistic
and dually atomistic.

3.2. **Theorem.** If $L \in \mathbb{D}$, then $\mathcal{S}_R(L)$ is dually semimodu-
lar if and only if either
(i) L is isomorphic to an ordinal sum $L' \oplus C \oplus L$, where L' and
L are (possibly empty) relatively complemented distributive lat-
tices and C is a chain or
(ii) all prime ideals of L are maximal, except possibly one,
or
(ii') all prime ideals of L are minimal, except possibly one.
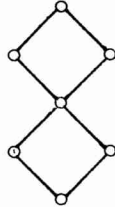Let L_7 be the 7-element lattice of figure 1.

Figure 1

Since the dual of L_7 is the ordinal sum of two 2-element antichains, Theorem 2 7 dualizes as follows.

3.3. <u>Theorem</u>. <u>If</u> L $\in$ $\mathbb{D}$, <u>then</u> $\mathcal{G}_R(L)$ <u>is dually geometric if and only if</u> L <u>is isomorphic to</u> $(B \oplus 1) \times B'$ <u>or to</u> $B \oplus C \oplus B'$, <u>where</u> B <u>and</u> B' <u>are finite Boolean algebras and</u> C <u>is a not empty chain</u>.

3.4. <u>Theorem</u>. <u>Let</u> L $\in$ $\mathbb{D}$.
a) <u>If</u> L <u>is not isomorphic to</u> L_7, <u>then the following assertions are equivalent</u>:

(i) $\mathcal{G}_R(L)$ <u>is modular</u>:

(ii) $\mathcal{G}_R(L)$ <u>is semimodular</u>;

(iii) $\mathcal{G}_R(L)$ <u>is geometric</u>;

(iv) <u>in</u> $\mathcal{G}_R(L)$, <u>the supremum of two atoms covers each of these atoms</u>;

(v)   L <u>is isomorphic to a sublattice of</u> $\underline{2}^2 \oplus C \oplus \underline{2}^2$ <u>(for some chain</u> C), <u>or to</u> $\underline{2}^3$ <u>or to</u> $\underline{2} \times \underline{3}$.

b) <u>If</u> L <u>is isomorphic to</u> L_7, <u>then (ii),(iii) and (iv) hold but</u> $\mathcal{G}_R(L)$ <u>is not modular</u>.

3.5. <u>Theorem</u>. <u>Let</u> L $\in$ $\mathbb{D}$. <u>Then the following are equivalent</u>:

(i) $\mathcal{G}_R(L)$ <u>is Boolean</u>;

(ii) $\mathscr{S}_R(L)$ is distributive;

(iii) $\mathscr{S}_R(L)$ is uniquely complemented;

(iv)   L is a chain or a four-element Boolean algebra.

We conclude by two corollaries of the above results which shed some light on the problem of the characterization of $\mathscr{S}_R(L)$. We are concerned here with the abstract characterization, but there is no difficulty to adapt our results to have information on the concrete characterization problem.

3.6. Theorem. Let $S \in \mathbb{D}$. Then S is isomorphic to $\mathscr{S}_R(L)$ for some $L \in \mathbb{D}$ if and only if S is a complete atomic Boolean lattice.

Proof. If $\mathscr{S}_R(L)$ is distributive, then it is Boolean, complete and atomic by 3.1 and 3.5.

Conversely, let C be a set such that S is isomorphic to the power set of C. Consider any linear ordering on C and define L to be C with supplementary bounds 0 and 1. Then $\mathscr{S}_R(L)$ is isomorphic to S.

3.7. Theorem. Let S be a modular lattice. Then S is isomorphic to $\mathscr{S}_R(L)$ for some L if and only if S is of one of the forms B, $B \times M_5$, or $B \times M_5 \times M_5$, where B is a complete atomic Boolean lattice.

Proof. Theorem 3.4 (and an easy computation) shows that the condition is necessary.

To prove that it is sufficient, let C be a bounded chain, given by 3.6, such that $\mathscr{S}_R(C)$ is isomorphic to B. Disregarding the case where B is trivial, we choose L to be C (resp. $C \oplus \underline{2}^2$, $\underline{2}^2 \oplus C \oplus \underline{2}^2$) and it follows that $\mathscr{S}_R(L)$ is isomorphic

to B (resp. $B \times M_5$, $B \times M_5 \times M_5$).

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