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HOW TO DEFINE REASONABLY WEIGHTED SOBOLEV SPACES

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Dedicated to the memory of Svatopluk FUČÍK

Abstract: The classes of weight functions are shown for which the corresponding weighted Sobolev space is guaranteed to be complete, i.e. a Banach space. Further, it is shown how to modify the definition of the weighted space if the weight functions do not belong to the class mentioned.

Key words: Linear function space, Banach space, Sobolev space, weighted Sobolev space, weight function.

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0. Introduction

Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$, let $\alpha \in (\mathbb{N}_0)^N$ be multi-indices of length $|\alpha| \leq k$ and S a set of weight functions w_α, $|\alpha| \leq k$, defined on an open set $\Omega \subset \mathbb{R}^N$. The Sobolev weight space

$$W^{k,p}(\Omega;S)$$

is usually defined as the (linear) set of all functions $u = u(x)$ on Ω such that $D^\alpha u \cdot w_\alpha^{1/p} \in L^p(\Omega)$ for $|\alpha| \leq k$.

For various reasons one needs $W^{k,p}(\Omega;S)$ to be a *normed* linear space and moreover, a *complete* normed space, i.e. a *Banach space*. However, in some cases (i.e. for some weight functions w_α) not even this is guaranteed, and therefore the question arises for what classes of weight functions w_α it is possible to define a weighted Sobolev space in such a way that it is a Banach space. Here, such classes are described.

1. Weights of class $B_p(\Omega)$

1.1. DEFINITION. Let Ω be an open set in $\mathbb{R}^N$. By the symbol

(1.1) $$\mathcal{W}(\Omega)$$

we denote the set of all measurable, a.e. in Ω positive and finite functions $w = w(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*.

1.2. DEFINITION. Let $\Omega \subset \mathbb{R}^N$, $p > 1$, $w \in \mathcal{W}(\Omega)$. By the symbol

(1.2) $$L^p(\Omega;w)$$

we denote the set of all measurable functions $u = u(x)$, $x \in \Omega$, such that

(1.3) $$||u||^p_{p,w,\Omega} = \int_\Omega |u(x)|^p \, w(x) \, dx < \infty .$$

For $w(x) \equiv 1$ we obtain the usual Lebesgue space $L^p(\Omega)$; in this case we write $||u||_{p,\Omega}$ instead of $||u||_{p,w,\Omega}$.

The following assertion is well known (see, e.g., [1], Theorem III. 6.6):

1.3. THEOREM. *The space* $L^p(\Omega;w)$ *equipped with the norm* $||\cdot||_{p,w,\Omega}$ *from* (1.3) *is a Banach space.*

1.4. DEFINITION. Let $p > 1$. We shall say that a weight function $w \in \mathcal{W}(\Omega)$ satisfies *condition* $B_p(\Omega)$ and write

(1.4) $$w \in B_p(\Omega) ,$$

if

(1.5) $$w^{-1/(p-1)} \in L^1_{loc}(\Omega) .$$

1.5. THEOREM. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set,* $p > 1$, $w \in B_p(\Omega)$, Q *a compact set in* $\mathbb{R}^N$, $Q \subset \Omega$. *Then*

(1.6) $$L^p(\Omega;w) \hookrightarrow L^1(Q) .$$

(Here $\hookrightarrow$ stands for a continuous imbedding.)

P r o o f : The assertion follows immediately from the Hölder inequality, since for $u \in L^p(\Omega;w)$ we have

$$\int_Q |u(x)|\,dx = \int_Q |u(x)|\, w^{1/p}(x)\, w^{-1/p}(x)\,dx \le$$

$$\le ||u||_{p,w,Q}\Big(\int_Q w^{-1/(p-1)}(x)\,dx\Big)^{\frac{p-1}{p}} \le c||u||_{p,w,\Omega}$$

with c independent of u

1.6. COROLLARY. Under the assumptions of Theorem 1.5 we have $L^p(\Omega;w) \subset L^1_{loc}(\Omega)$. Using the usual identification of a *regular* distribution from $\mathcal{D}'(\Omega)$ with a function from $L^1_{loc}(\Omega)$ we conclude that

$$L^p(\Omega;w) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega) \tag{1.7}$$

for $w \in B_p(\Omega)$ Therefore, for functions $u \in L^p(\Omega;w)$ with $w \in B_p(\Omega)$, the *distributional derivatives* $D^\alpha u$ of u have sense.

If $w \notin B_p(\Omega)$, then the inclusion (1.7) need not hold. This follows from

1.7. EXAMPLE. For $N = 1$, $\Omega = (-1/2,1/2)$, $p > 1$ and $w(x) = |x|^{p-1}$ we have $w \notin B_p(\Omega)$ since $w^{-1/(p-1)}(x) = |x|^{-1}$. Let us take $u(x) = |x|^{-1}|\ln|x||^\lambda$ with $\lambda \in (-1,-1/p)$. Then

$$||u||^p_{p,w,\Omega} = \int_{-1/2}^{1/2} |x|^{-p}|\ln|x||^{\lambda p}|x|^{p-1}\,dx = 2\int_0^{1/2} |x|^{-1}|\ln x|^{\lambda p}\,dx =$$

$$= 2\int_{\ln 2}^{\infty} t^{\lambda p}\,dt < \infty$$ since $\lambda < -1/p$, i.e. $\lambda p < -1$, and consequently, $u \in L^p(\Omega;w)$. On the other hand, $u \notin L^1_{loc}(\Omega)$ since we have $\lambda > -1$ and so, $\int_{-1/4}^{1/4} |u(x)|\,dx = 2\int_{\ln 4}^{\infty} t^\lambda\,dt = \infty$.

1.8. REMARK. Let $w \in B_p(\Omega)$, $\phi \in C_0^\infty(\Omega)$ ($= \mathcal{D}(\Omega)$) and let a multi-index $\gamma \in (\mathbb{N}_0)^N$ be fixed. Then the formula

$$L_\gamma(u) = \int_\Omega u D^\gamma \phi\,dx\ , \quad u \in L^p(\Omega;w)\ , \tag{1.8}$$

defines a continuous linear functional L_γ on $L^p(\Omega;w)$. Indeed, if we denote $Q = \operatorname{supp}\phi$, then $Q = \overline{Q} \subset \Omega$ and the Hölder inequality implies

$$|L_\gamma(u)| \leq \int_\Omega |u| w^{1/p} |D^\gamma \phi| w^{-1/p}\, dx \leq$$

$$\leq ||u||_{p,w,\Omega} \left(\int_Q |D^\gamma \phi|^{p/(p-1)} w^{-1/(p-1)}\, dx \right)^{\frac{p-1}{p}} \leq$$

$$\leq ||u||_{p,w,\Omega} \cdot \max_Q |D^\gamma \phi| \cdot \left(\int_Q w^{-1/(p-1)}\, dx \right)^{\frac{p-1}{p}} ;$$

here, the last integral is finite in view of (1.5).

1.9. PRELIMINARY DEFINITION OF THE WEIGHTED SOBOLEV SPACE. Let $\Omega \subset \mathbb{R}^N$ be an open set, $p > 1$. Let $\mathcal{M}_1$ be a nonempty set of multi-indices of length 1 and let $\mathcal{M} = \{\theta\} \cup \mathcal{M}_1$ with $\theta = (0,0,\ldots,0)$. Denote $S = \{w_\alpha \in \mathcal{W}(\Omega)\ ,\ \alpha \in \mathcal{M}\}$ and let us define the *Sobolev space with weight* S,

$$W^{1,p}(\Omega;S) ,$$

as the set of all functions $u \in L^p(\Omega;w_\theta) \cap L^1_{loc}(\Omega)$ such that their distributional derivatives $D^\alpha u$ with $\alpha \in \mathcal{M}_1$ are again elements of $L^p(\Omega;w_\alpha) \cap L^1_{loc}(\Omega)$ (i.e., $D^\alpha u$ are *regular* distributions).

The expression

$$(1.9) \qquad ||u||_{1,p,S,\Omega} = \left(\sum_{\alpha \in \mathcal{M}} ||D^\alpha u||^p_{p,w_\alpha,\Omega} \right)^{1/p}$$

obviously is a norm on the linear space $W^{1,p}(\Omega;S)$.

1.10. REMARK. If certain of the weight functions w_α satisfy the condition $B_p(\Omega)$, then the assumption $D^\alpha u \in L^p(\Omega;w_\alpha) \cap L^1_{loc}(\Omega)$ in Definition 1.9 can be replaced in view of (1.7) by the assumption

$$(1.10) \qquad D^\alpha u \in L^p(\Omega;w_\alpha) .$$

1.11. THEOREM. *Let* $w_\alpha \in B_p(\Omega)$ *for all* $\alpha \in \mathcal{M}$. *Then* $W^{1,p}(\Omega;S)$ *is a Banach space if equipped with the norm* (1.9).

P r o o f : Let $\{u_n\}$ be a Cauchy sequence in $W^{1,p}(\Omega;S)$. Then $\{D^\alpha u_n\}$ is a Cauchy sequence in $L^p(\Omega;w_\alpha)$ for every $\alpha \in \mathcal{M}$, and by Theorem 1.3 there exist functions $u_\alpha \in L^p(\Omega;w_\alpha)$, $u_\alpha = \lim_{n\to\infty} D^\alpha u_n$

in $L^p(\Omega;w_\alpha)$.

For a fixed $\alpha \in \mathcal{M}_1$ and a fixed $\phi \in C_0^\infty(\Omega)$, let us consider the functional L_α from (1.8). It is a continuous linear functional on $L^p(\Omega;w_\Theta)$, and consequently,

$$L_\alpha(u_n) \longrightarrow L_\alpha(u_\Theta) \quad \text{for} \quad n \longrightarrow \infty .$$

At the same time, $L_\Theta(v)$ defines a continuous linear functional on $L^p(\Omega;w_\alpha)$, and consequently,

$$L_\Theta(D^\alpha u_n) \longrightarrow L_\Theta(u_\alpha) \quad \text{for} \quad n \longrightarrow \infty .$$

By the definition of the distributional derivative we have $L_\alpha(u_n) = - L_\Theta(D^\alpha u_n)$ (notice that $|\alpha| = 1$), and by a limiting process this formula yields

$$L_\alpha(u_\Theta) = - L_\Theta(u_\alpha) .$$

This relation holds for every $\phi \in C_0^\infty(\Omega)$ and therefore, u_α is the distributional derivative of u_Θ :

$$u_\alpha = D^\alpha u_\Theta .$$

Since $D^\alpha u_\Theta \in L^p(\Omega;w_\alpha) = L^p(\Omega;w_\alpha) \cap L^1_{loc}(\Omega)$, we have $u_\Theta \in$ $\in W^{1,p}(\Omega;S)$ and

$$||u_n - u_\Theta||^p_{1,p,S,\Omega} = \sum_{\alpha \in \mathcal{M}} ||D^\alpha u_n - D^\alpha u_\Theta||^p_{p,w_\alpha,\Omega} =$$

$$= \sum_{\alpha \in \mathcal{M}} ||D^\alpha u_n - u_\alpha||^p_{p,w_\alpha,\Omega} \longrightarrow 0$$

for $n \longrightarrow \infty$. Hence the Cauchy sequence $\{u_n\}$ converges to u_Θ in $W^{1,p}(\Omega;S)$, i.e., $W^{1,p}(\Omega;S)$ is complete.

The condition $w_\alpha \in B_p(\Omega)$ in Theorem 1.11 is essential. This follows from

<u>1.12. EXAMPLE</u>. Let us take $N = 1$, $\Omega = (-1,1)$, $p = 2$, $\mathcal{M} =$ $= \{0,1\}$, $\lambda, \mu \in \mathbb{R}$ and $S = \{w_0(x) = |x|^\lambda, \ w_1(x) = |x|^\mu\}$. Obviously $w_0, w_1 \in \mathcal{W}(\Omega)$, but for $\lambda \geq 1$, $\mu \geq 1$ we have $w_0 \notin B_2(\Omega)$, $w_1 \notin B_2(\Omega)$. The space $W^{1,2}(\Omega;S)$ is noncomplete if the parameters λ, μ are suitably chosen - we will show this by constructing a Cauchy sequence $\{u_n\}$ in $W^{1,2}(\Omega;S)$ which has no limit in this space.

For this purpose, let us consider the function

(1.11) $$u(x) = \begin{cases} 0 & \text{for } x \leq 0 , \\ x^{\gamma} & \text{for } x > 0 \end{cases}$$

with $\gamma \in \mathbb{R}$. If

(1.12) $$\gamma \leq -1$$

then

(1.13) $$u \notin L^1_{loc}(\Omega) .$$

Let us denote $\Omega_1 = (-1,0)$, $\Omega_2 = (0,1)$. Obviously $u \in W^{1,2}(\Omega_1;S)$ and $||u||_{1,2,S,\Omega_1} = 0$. Further, $u \in W^{1,2}(\Omega_2;S)$ if and only if

(1.14) $$\gamma > -\frac{1}{2} - \frac{\lambda}{2} \quad \text{and} \quad \gamma > \frac{1}{2} - \frac{\mu}{2} ,$$

since $||u||^2_{1,2,S,\Omega_2} = \int_0^1 x^{2\gamma} x^{\lambda} \, dx + \gamma^2 \int_0^1 x^{2\gamma-2} x^{\mu} \, dx$.

For $\delta \in (0,1)$ we define

$$g_{\delta}(x) = \begin{cases} 0 & \text{for } x \in (-1, \delta/2\rangle , \\ \frac{2}{\delta} x - 1 & \text{for } x \in (\delta/2, \delta) , \\ 1 & \text{for } x \in \langle\delta, 1) \end{cases}$$

and denote

$$v_{\delta}(x) = u(x) g_{\delta}(x) .$$

Easy computation shows that if (1.14) holds then

(1.15) $$\lim_{\delta \to 0} ||u - v_{\delta}||_{1,2,S,\Omega_i} = 0 \quad \text{for } i = 1,2 .$$

If we denote $u_n = v_{1/n}$, then evidently $u_n \in W^{1,2}(\Omega;S)$ and (1.15) implies that $\{u_n\}$ is a Cauchy sequence in $W^{1,2}(\Omega_i;S)$ for both $i = 1$ and $i = 2$. But then $\{u_n\}$ is a Cauchy sequence in $W^{1,2}(\Omega;S)$, too, since $||v||^2_{1,2,S,\Omega} = ||v||^2_{1,2,S,\Omega_1} + ||v||^2_{1,2,S,\Omega_2}$.

Let us suppose that $W^{1,2}(\Omega;S)$ is *complete*. Then there exists an element $u^* \in W^{1,2}(\Omega;S)$ such that

(1.16) $$||u^* - u_n||_{1,2,S,\Omega} \to 0 \quad \text{for } n \to \infty .$$

A fortiori, this relation takes place if we replace Ω by Ω_i , and then in view of (1.15) we have $u = u^*$ a.e. in Ω_i , $i = 1,2$, i.e. $u = u^*$ a.e. in Ω . The function u^* belongs to $W^{1,2}(\Omega;S)$ and therefore $u^* \in L^1_{loc}(\Omega)$. Hence also $u \in L^1_{loc}(\Omega)$ and this leads to a *contradiction* with (1.13). Since the conditions (1.12) and (1.14) can be satisfied e.g. by choosing $\lambda = 2$, $\mu = 4$ and $\gamma \in (-3/2,-1\rangle$, the contradiction mentioned shows that in this case the space $W^{1,2}(\Omega;S)$ is not complete.

1.13. REMARK. In Example 1.12 we have constructed, for a given function $u \in W^{1,2}(\Omega;S)$, a Cauchy sequence $\{u_n\} \subset W^{1,2}(\Omega;S)$ which approximates u in both $W^{1,2}(\Omega_1;S)$ and $W^{1,2}(\Omega_2;S)$. Let us mention that we can choose another sequence $\{u_n^*\}$ with the same properties but, moreover, such that $u_n^* \in C^\infty(\overline{\Omega})$. This can be done by using the imbedding

$$W^{1,2}(\Omega) \hookrightarrow W^{1,2}(\Omega;S)$$

which holds for $\lambda \geq 0$, $\mu \geq 0$, and the facts that the above functions u_n belong even to $W^{1,2}(\Omega)$ and that $C^\infty(\overline{\Omega})$ is dense in $W^{1,2}(\Omega)$.

2. Weakening the conditions on w_α

In Theorem 1.11 we have assumed that $w_\alpha \in B_p(\Omega)$ for *all* $\alpha \in \mathcal{M}$. These conditions can be weakened, namely, we can omit this assumption for $\alpha = \theta$.

2.1. THEOREM. *Let* $p > 1$, $w_\alpha \in B_p(\Omega)$ *for* $\alpha \in \mathcal{M}_1$, $w_\theta \in \mathcal{W}(\Omega)$. *Then* $W^{1,p}(\Omega;S)$ *is a Banach space if equipped with the norm* (1.9).

2.2. REMARK. For the proof of Theorem 2.1 we need a "one-dimensional" lemma. To this end, let us introduce - for an open set $G \subset \mathbb{R}^N$ - the set

$$AC(G)$$

of all functions *absolutely continuous* on every compact interval $\overline{I} \subset G$. Every function $u \in AC(G)$ has a derivative a.e. in G ; we will denote it by

$$\dot{u} .$$

2.3. LEMMA. *Let* G *be an open set in* $\mathbb{R}$, $\mathcal{M} = \{0,1\}$, $S =$

$= \{w_0, w_1\}$, $p > 1$. *Let*

$$w_1 \in B_p(G)\ , \quad w_0 \in W(G) \tag{2.1}$$

and let $\{u_n\}$ *be a Cauchy sequence in* $W^{1,p}(G;S)$ *such that*

$$\begin{aligned} u_n &\to u^0 \quad \text{in} \quad L^p(G;w_0)\ , \\ u_n' &\to u^1 \quad \text{in} \quad L^p(G;w_1)\ . \end{aligned} \tag{2.2}$$

Then the function u^0 *can be changed on a set of zero measure so that it satisfies*

$$u^0 \in AC(G)\ , \tag{2.3}$$

$$\dot{u}^0 = u^1\ . \tag{2.4}$$

P r o o f : Since for an open set $G \subset \mathbb{R}$ we have $G = \bigcup_{j=1}^{\infty} I_j$ where I_j are disjoint open intervals, we can assume that G is an (open) interval. It follows from the definition of the space $W^{1,p}(G;S)$ that $u_n \in W^{1,p}(G;S)$ implies $u_n \in W^{1,1}_{loc}(G)$. So we can change u_n on a set of zero measure in such a way that $u_n \in AC(G)$ and that the derivative $\dot{u}_n$ coincides a.e. in G with the distributional derivative u_n' of u_n (see, e.g., [2], Theorem 5.6.3).

From (2.2) it follows that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that, for $n \to \infty$,

$$\begin{aligned} u_{n_k} &\to u^0 \quad \text{a.e. in } G\ , \\ u_{n_k}' &\to u^1 \quad \text{a.e. in } G\ . \end{aligned} \tag{2.5}$$

Now let $\bar{x}$ be a point from G such that

$$u_{n_k}(\bar{x}) \to u^0(\bar{x}) \quad \text{for } n \to \infty\ . \tag{2.6}$$

Since $u_{n_k} \in AC(G)$, we have

$$u_{n_k}(x) = \int_{\bar{x}}^{x} u_{n_k}'(t)\,dt + u_{n_k}(\bar{x}) \tag{2.7}$$

for $x \in G$. Let us define the function u^* on G by

$$u^*(x) = \int_{\bar{x}}^{x} u^1(t)\,dt + u^0(\bar{x})\ . \tag{2.8}$$

Since $w_1 \in B_p(G)$, in view of (2.2) and of Corollary 1.6 we have $u^1 \in L^1_{loc}(G)$, and consequently, $u^* \in AC(G)$, too.

In view of (2.7) and (2.8) we have, for every $x \in G$,

(2.9) $$|u_{n_k}(x) - u^*(x)| \leq$$

$$\leq |u_{n_k}(\bar{x}) - u^0(\bar{x})| + \left|\int_{\bar{x}}^{x} |u'_{n_k}(t) - u^1(t)|\, dt\right| .$$

Since the closed interval with the endpoints x and $\bar{x}$ is contained in G , by virtue of Theorem 1.5 it follows from the second relation in (2.2) that the last term in (2.9) tends to zero for $n_k \to \infty$. This combined with (2.6) yields, in view of (2.9), that

$$u_{n_k}(x) \to u^*(x) \ , \quad x \in G$$

(x is arbitrary but fixed), and this together with (2.5) implies

$$u^0(x) = u^*(x) \quad \text{a.e. in } G .$$

If we change u^0 in such a way that $u^0(x) = u^*(x)$ for all $x \in G$, then we have (2.3) since $u^* \in AC(G)$.

Further, for the derivatives we have

$$\dot{u}^0 = \dot{u}^* = u^1 ,$$

the last equality being a consequence of (2.8), and hence (2.4) is proved as well.

2.4. PROOF OF THEOREM 2.1. Let $\{u_n\}$ be a Cauchy sequence in $W^{1,p}(\Omega;S)$. Then for every $\alpha \in \mathcal{M}$ there exists a function u_α ,

(2.10) $$u_\alpha = \lim_{n\to\infty} D^\alpha u_n \quad \text{in } L^p(\Omega;w_\alpha) .$$

For $\alpha \in \mathcal{M}_1$ we have, moreover, $u_\alpha \in L^1_{loc}(\Omega)$. It remains to prove that

(2.11) $$u_\Theta \in L^1_{loc}(\Omega) ,$$

(2.12) $$u_\alpha = D^\alpha u_\Theta .$$

For $i \in \{1,2,\dots,N\}$ let us write $x = (x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_N) \in \mathbb{R}^N$ in the form $x = (x'_i,x_i)$ with $x'_i \in \mathbb{R}^{N-1}$. If H is an open set in $\mathbb{R}^N$, then we denote by $P_i(H)$ the projection of H

onto the hyperplane $x_i = 0$ and by $H(x_i')$ the cut

$$H(x_i') = \{t \in \mathbb{R},\ (x_i',t) \in H\}\ ,\quad x_i' \in P_i(H)\ .$$

Let $\alpha \in \mathbb{M}_1$ be *fixed*, so that $\alpha = (0,\dots,0,1,0,\dots,0)$ with the component one on the i-th place. If we denote

$$\text{(2.13)} \qquad f_n^\alpha(x_i') = \int_{\Omega(x_i')} |D^\alpha u_n(x_i',t) - u_\alpha(x_i',t)|^p\, w_\alpha(x_i',t)\, dt$$

for $x_i' \in P_i(\Omega)$ then we can rewrite (2.10) as follows:

$$\text{(2.14)} \qquad \int_\Omega |D^\alpha u_n(x) - u_\alpha(x)|^p\, w_\alpha(x)\, dx = \int_{P_i(\Omega)} f_n^\alpha(x_i')\, dx_i' \longrightarrow 0\ ,$$

i.e. $f_n^\alpha(x_i') \longrightarrow 0$ in $L^1(P_i(\Omega))$. Therefore, there exists a subsequence $\{f_{n_k}^\alpha\}$ such that $f_{n_k}^\alpha(x_i') \longrightarrow 0$ a.e. in $P_i(\Omega)$, and in view of (2.13), this implies that for a.e. $x_i' \in P_i(\Omega)$,

$$D^\alpha u_{n_k}(x_i',\cdot) \longrightarrow u_\alpha(x_i',\cdot) \quad \text{in} \quad L^p(\Omega(x_i');\ w_\alpha(x_i',\cdot))\ .$$

This relation also holds if we replace α by θ.

For a fixed $x_i' \in P_i(\Omega)$, let us denote $u^0(t) = u_\theta(x_i',t)$, $u^1(t) = u_\alpha(x_i',t)$, $t \in \Omega(x_i') \subset \mathbb{R}$. Using Lemma 2.3, we can change u^0 on a set of zero measure so that $u^0 \in AC(\Omega(x_i'))$ and

$$\text{(2.15)} \qquad \dot{u}^0 = D^\alpha u_\theta(x_i',\cdot) = u^1 = u_\alpha(x_i',\cdot)$$

(note that D^α means $\partial/\partial x_i$).

Now, let Q be an arbitrary but fixed bounded open set, $Q \subset \overline{Q} \subset \Omega$, let $\phi = \phi_0\phi_1$ with $\phi_0 \in C_0^\infty(\Omega)$ such that $\phi_0(x) \equiv 1$ on Q and $\phi_1(x) = x_i$ [so that $\phi \in C_0^\infty(\Omega)$ and

$$\text{(2.16)} \qquad D^\alpha\phi(x) \equiv 1 \quad \text{for} \quad x \in Q\,].$$

Further, let H be an open set with $\operatorname{diam} H < \infty$ and such that $\operatorname{supp}\phi \subset H \subset \overline{H} \subset \Omega$. Then we have

$$\int_Q |u_\theta|\, dx = \int_Q |u_\theta|\cdot D^\alpha\phi\, dx \leqq \int_H |u_\theta|\cdot|D^\alpha\phi|\, dx =$$

$$= \int_{P_i(H)} \left(\int_{H(x_i')} |u_\theta(x_i',t)|\cdot|D^\alpha\phi(x_i',t)|\, dt \right) dx_i' =$$

$$= - \int_{P_i(H)} \left(\int_{H(x_i')} D^\alpha |u_\Theta(x_i',t)| \int_{-\infty}^{t} |D^\alpha \phi(x_i',\tau)| \, d\tau \, dt \right) dx_i' \le$$

$$\le \int_{P_i(H)} \left(\int_{H(x_i')} \left| D^\alpha |u_\Theta(x_i',t)| \right| \operatorname{diam} H \cdot \max_{\overline{H}} |D^\alpha \phi| \, dt \right) dx_i' \le$$

$$\le c_0 \int_{P_i(H)} \left(\int_{H(x_i')} |D^\alpha u_\Theta(x_i',t)| \, dt \right) dx_i' = c_0 \int_H |D^\alpha u_\Theta| \, dx$$

with $c_0 = \operatorname{diam} H \cdot \max_{\overline{H}} |D^\alpha \phi|$. In view of (2.15),

$$\int_Q |u_\Theta| \, dx \le c_0 \int_H |u_\alpha| \, dx < \infty$$

since $u_\alpha \in L^1_{loc}(\Omega)$. However, this implies that $u_\Theta \in L^1_{loc}(\Omega)$ and so, (2.11) is proved.

Finally, (2.12) follows from the fact that, for every $\phi \in C_0^\infty(\Omega)$, (2.15) and (2.11) imply

$$\int_\Omega u_\Theta D^\alpha \phi \, dx = \int_{P_i(\Omega)} \left(\int_{\Omega(x_i')} u_\Theta(x_i',t) D^\alpha \phi(x_i',t) \, dt \right) dx_i' = \tag{2.17}$$

$$= - \int_{P_i(\Omega)} \left(\int_{\Omega(x_i')} D^\alpha u_\Theta(x_i',t) \phi(x_i',t) \, dt \right) dx_i' =$$

$$= - \int_{P_i(\Omega)} \left(\int_{\Omega(x_i')} u_\alpha(x_i',t) \phi(x_i',t) \, dt \right) dx_i' = - \int_\Omega u_\alpha \phi \, dx .$$

The following example shows that the condition $w_\alpha \in B_p(\Omega)$ *cannot be omitted* for $\alpha \in \mathcal{M}_1$:

2.5. EXAMPLE. Let us consider the space $W^{1,2}(\Omega;S)$ from Example 1.12, but now with $\lambda \in \langle 0,1)$, $\mu \ge 1$. Then $w_0 \in B_2(\Omega)$. $w_1 \notin B_2(\Omega)$. Let us consider the function $u(x)$ from (1.11) with $\gamma = 0$, i.e.

$$u(x) = 0 \text{ for } x \le 0 , \quad u(x) = 1 \text{ for } x > 0 . \tag{2.18}$$

Let us mention that now the conditions (1.14) have the form

$$\lambda > -1 , \quad \mu > 1 . \tag{2.19}$$

Proceeding analogously as in Example 1.12 we construct a Cauchy sequence $\{u_n\}$ in $W^{1,2}(\Omega;S)$, and the assumption of completeness of this space implies the existence of the function $u^* = \lim_{n\to\infty} u_n$ in $W^{1,2}(\Omega;S)$ such that $u = u^*$ a.e. in Ω. But this leads to a *contradiction*: The function u^* as an element of $W^{1,2}(\Omega;S)$ has the distributional derivative $(u^*)' \in L^1_{loc}(\Omega) \cap L^2(\Omega;w_1)$. However, the distributional derivative of the function u from (2.18) does not belong to $L^1_{loc}(\Omega)$, since it is the *Dirac distribution*.

Consequently, the space $W^{1,2}(\Omega;S)$ *is not complete* for

(2.20) $\lambda \in \langle 0,1)$, $\mu \in (1,\infty)$.

2.6. EXAMPLE. All the foregoing examples have been one-dimensional, but it is easy to construct more-dimensional examples. For instance, if we take $p = 2$, $N = 2$, $\Omega = (-1,1) \times (-1,1)$, $\mathcal{M} = \{(0,0),(1,0),(0,1)\}$, $w_{(0,0)}(x,y) = |x|^\lambda$, $w_{(1,0)}(x,y) = |x|^\mu$, $w_{(0,1)}(x,y) \equiv 1$, with λ, μ from (2.20), then $w_{(0,0)}, w_{(0,1)} \in B_2(\Omega)$, $w_{(1,0)} \notin B_2(\Omega)$ and the space $W^{1,2}(\Omega;S)$ *is not complete* since the distributional derivative $\partial u/\partial x$ of the function u defined by $u(x,y) \equiv 0$ if $x \in (-1,0\rangle$, $u(x,y) \equiv 1$ if $x \in (0,1)$, *is not a regular distribution*, so that $\partial u/\partial x \notin L^1_{loc}(\Omega)$.

3. Exceptional sets Definition of the weighted Sobolev space

In Example 2.6, the "bad" set which causes the noncompleteness of $W^{1,2}(\Omega;S)$ was the open segment $\{[0,y];\ -1 < y < 1\}$. So, we are led to

3.1. DEFINITION. Let $w \in \mathcal{W}(\Omega)$, $p > 1$ and denote

$$(3.1)\qquad M_p(w) = \{x \in \Omega;\ \int_{\Omega\cap U(x)} w^{-1/(p-1)}(y)\,dy = \infty \text{ for every neighbourhood } U(x) \text{ of } x\}.$$

Obviously, $M_p(w) = \emptyset$ for $w \in B_p(\Omega)$. Now, we have

3.2. LEMMA. *Let* $\Omega \subset \mathbb{R}^N$ *be open,* $p > 1$, $w \in \mathcal{W}(\Omega)$, $w \notin B_p(\Omega)$. *Then*

(i) $M_p(w)$ *is a nonempty closed set in* Ω ;

(ii) $w \in B_p(\Omega - M_p(w))$.

P r o o f : Let us denote $M = M_p(w)$. Definition 3.1 implies that if $x \in \Omega - M$, then there exists a ball $U(x,\varepsilon) = \{z \in \mathbb{R}^N; |x - z| < \varepsilon\}$ such that

$$\int_{\Omega \cap U(x,\varepsilon)} w^{-1/(p-1)}(y)\,dy < \infty . \tag{3.2}$$

Now, let K be a compact set, $K \subset \Omega - M$. The system of all balls $U(x,\varepsilon)$ from (3.2) with $x \in K$ forms an open covering of K ; from this covering we select a finite covering $\bigcup_{i=1}^{m} U_i$, and since

$$\int_K w^{-1/(p-1)}(y)\,dy \leqq \sum_{i=1}^{m} \int_{U_i} w^{-1/(p-1)}(y)\,dy < \infty ,$$

we have $w^{-1/(p-1)} \in L^1(K)$. The set $K \subset \Omega - M$ was arbitrary, which proves the assertion (ii).

The assumption $M = \emptyset$ implies by (ii) that $w \in B_p(\Omega)$, and this contradicts the assumption $w \notin B_p(\Omega)$. Consequently, M is nonempty. Further, let $x \in \Omega - M$. Then there exists a neighbourhood $U(x)$ such that $U(x) \subset \Omega - M$, namely, the ball $U(x,\varepsilon)$ from (3.2) with $\varepsilon > 0$ such that $U(x,\varepsilon) \subset \Omega$: Indeed, for every $z \in U(x,\varepsilon)$ we have $z \in \Omega - M$ since

$$\int_{\Omega \cap U(z,\delta)} w^{-1/(p-1)}(y)\,dy < \infty$$

for $\delta = \varepsilon - |x - z|$. Consequently, $\Omega - M$ is open in $\mathbb{R}^N$ and the assertion (i) is proved, too.

Another property of the exceptional set $M_p(w)$ is described by the following theorem, whose proof is left to the reader:

3.3. THEOREM. *Let* $\Omega \subset \mathbb{R}^N$ *be open,* $p > 1$ *Let* $w \in \mathcal{W}(\Omega)$ *be continuous a.e. in* Ω . *Then*

$$\operatorname{meas}(M_p(w)) = 0 . \tag{3.3}$$

3.4. REMARK. Let $W^{1,p}(\Omega;S)$ be the space from Definition 1.9. Let us denote

(3.4) $$\mathcal{B} = \bigcup_{\substack{\alpha \in \mathfrak{M}_1 \\ w_\alpha \notin B_p(\Omega)}} M_p(w_\alpha)$$

with $M_p(w_\alpha)$ from (3.1). Theorem 1.11 implies that if $\mathcal{B} = \emptyset$, then $W^{1,p}(\Omega;S)$ is a Banach space, while Examples 2.5 and 2.6 indicate that if $\mathcal{B} \neq \emptyset$, then $W^{1,p}(\Omega;S)$ need not be complete. Therefore, we are led to the following definition of the weighted Sobolev space:

3.5. DEFINITION. Let Ω, p, $\mathfrak{M}_1$, $\mathfrak{M}$ and S be as in Definition 1.9, with $w_\alpha \in \mathcal{W}(\Omega)$ for $\alpha \in \mathfrak{M}$. Let $\mathcal{B}$ be the set from (3.4). Then we define the *Sobolev space with weight* S,

$$W^{1,p}(\Omega;S) ,$$

as the space $W^{1,p}(\Omega - \mathcal{B}; S)$, considered in the sense of Definition 1.9.

It follows from the assertion (ii) of Lemma 3.2 that $w_\alpha \in B_p(\Omega - \mathcal{B})$ for every $\alpha \in \mathfrak{M}_1$. Hence, by Theorem 2.1, the space $W^{1,p}(\Omega - \mathcal{B}; S)$ and, consequently, the space $W^{1,p}(\Omega;S)$ (in the sense of Definition 3.5) is *complete*, i.e. a Banach space.

3.6. REMARK. Another way how to guarantee the *completeness* of the weighted Sobolev space is to define it as the *completion* of the set $W^{1,p}(\Omega;S)$ from Definition 1.9 with respect to the norm (1.9). However, in this case the completion could contain nonregular distributions (see Example 1.12) or functions whose distributional derivatives are not regular distributions (see Examples 2.5, 2.6, which indicate that it is the set $\mathcal{B} \subset \Omega$ which makes difficulties). Therefore, Definition 3.5 seems to be more natural.

4. Concluding remarks. The space $W_0^{1,p}(\Omega;S)$

4.1. REMARK. The classical Sobolev space $W^{1,p}(\Omega)$ is often defined (for a "reasonable" domain Ω) as the closure of the set $C^\infty(\bar{\Omega})$ in the corresponding norm $||\cdot||_{1,p,\Omega}$. If we want to proceed analogously in the case of weighted spaces, we need first of all that

(4.1) $\qquad C^{\infty}(\overline{\Omega}) \subset W^{1,p}(\Omega;S)$

hold. This relation excludes a great number of weight functions, e.g. weights of the type $|x - x_0|^{-\lambda}$ with large $\lambda > 0$, $x_0 \in \overline{\Omega}$ fixed. On the other hand, such weight functions evidently belong to the class $B_p(\Omega)$.

At the same time, Remark 1.13 shows that even if the condition (4.1) is fulfilled, the completion could lead to spaces as in Remark 3.6, i.e. to Banach spaces with elements which are nonregular distributions.

4.2. THE SPACE $W_0^{1,p}(\Omega;S)$. In various applications, in particular, for the investigation of the Dirichlet problem for elliptic partial differential equations, we need the space $W_0^{1,p}(\Omega;S)$ defined as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm (1.9). In order to be able to introduce this space, we need the inclusion

(4.2) $\qquad C_0^{\infty}(\Omega) \subset W^{1,p}(\Omega;S)$

which is evidently fulfilled if

(4.3) $\qquad w_\alpha \in L^1_{loc}(\Omega)$ for all $\alpha \in \mathfrak{M}$.

Hence we are able to introduce the following

4.3. PRELIMINARY DEFINITION OF THE SPACE $W_0^{1,p}(\Omega;S)$. Let Ω, p, $\mathfrak{M}_1$, $\mathfrak{M}$ and S be as in Definition 1.9. Let $w_\alpha \in B_p(\Omega)$ for $\alpha \in \mathfrak{M}_1$ and $w_\alpha \in L^1_{loc}(\Omega)$ for $\alpha \in \mathfrak{M}$. Then we define

(4.4) $\qquad W_0^{1,p}(\Omega;S) = \overline{C_0^{\infty}(\Omega)}$,

the closure being taken with respect to the norm $||\cdot||_{1,p,S,\Omega}$ from (1.9).

Condition (4.3) is not only sufficient but also necessary for (4.2):

4.4 LEMMA. *The inclusion* (4.2) *is fulfilled if and only if* (4.3) *holds.*

P r o o f : If the condition (4.3) holds then evidently (4.2) is fulfilled. Conversely, let us suppose that (4.2) holds. Let $\alpha \in \mathfrak{M}$ and let $Q \subset \Omega$ be a compact set. Then there exists a function

$\phi \in C_0^\infty(\Omega)$ such that $D^\alpha\phi(x) \equiv 1$ for $x \in Q$ (see (2.16)), and this identity together with (4.2) implies

$$0 \leq \int_Q w_\alpha\, dx = \int_Q |D^\alpha\phi|^p w_\alpha\, dx \leq \int_\Omega |D^\alpha\phi|^p w_\alpha\, dx \leq ||\phi||^p_{1,p,S,\Omega} < \infty$$

and consequently, $w_\alpha \in L^1_{loc}(\Omega)$.

It follows from Lemma 4.4 that for some weights S the inclusion (4.2) need not hold. So we are led to the notion of another exceptional set:

4.5. DEFINITION. Let $w \in \mathcal{W}(\Omega)$ and denote

$$M_0(w) = \Big\{x \in \Omega;\ \int_{\Omega\cap U(x)} w(t)\,dt = \infty \text{ for every neighbourhood } U(x) \text{ of } x\Big\}\ . \tag{4.5}$$

This set is an analogue of the set $M_p(w)$ from Definition 3.2 (we obtain it formally by setting $p = 0$ in (3.1)). Obviously $M_0(w) = \emptyset$ for $w \in L^1_{loc}(\Omega)$. Similarly to Lemma 3.3 and Theorem 3.4, we have

4.6. LEMMA. *Let* $\Omega \subset \mathbb{R}^N$ *be open,* $w \in \mathcal{W}(\Omega)$ *and* $w \notin L^1_{loc}(\Omega)$. *Then*

(i) $M_0(w)$ *is a nonempty closed set in* Ω ,

(ii) $w \in L^1_{loc}\big(\Omega - M_0(w)\big)$.

If w *is continuous a.e. in* Ω *, then*

$$\operatorname{meas}\big(M_0(w)\big) = 0\ .$$

Now, we are able to introduce the definition of the weighted Sobolev space $W_0^{1,p}(\Omega;S)$:

4.7. DEFINITION. Let Ω , p , $\mathcal{M}_1$, $\mathcal{M}$ and S be as in Definition 1.9. Denoting

$$C = \bigcup_{\substack{\alpha\in\mathcal{M} \\ w_\alpha \notin L^1_{loc}(\Omega)}} M_0(w_\alpha) \tag{4.6}$$

with $M_0(w_\alpha)$ from (4.5) we define

$$W_0^{1,p}(\Omega;S) = \overline{V} \tag{4.7}$$

where

$$V = \{f;\ f = g|_{\Omega - \mathcal{B}}\ ,\quad g \in C_0^\infty(\Omega - \mathcal{C})\}$$

with $\mathcal{B}$ from (3.11), the closure in (4.7) being taken with respect to the norm $||\cdot||_{1,p,S,\Omega}$ from (1.9).

Thus we have obtained again a Banach space, which is a subspace of $W^{1,p}(\Omega;S)$.

4.8. REMARK. If $w_\alpha \in B_p(\Omega)$ for every $\alpha \in \mathcal{M}_1$, then $\mathcal{B} = \emptyset$ and (4.7) yields

$$W_0^{1,p}(\Omega;S) = \overline{C_0^\infty(\Omega - \mathcal{C})}\ .$$

If we suppose in addition that $w_\alpha \in L^1_{loc}(\Omega)$ for every $\alpha \in \mathcal{M}$, then $\mathcal{C} = \emptyset$ and the space $W_0^{1,p}(\Omega;S)$ from (4.7) coincides with the space $W_0^{1,p}(\Omega;S)$ from (4.4).

4.9. REMARK. For various purposes, in particular in weighted inequalities for maximal functions and other (integral) operators, the class A_p of weight functions introduced by B. MUCKENHOUPT [3] plays an important role. Here $w \in A_p$ means that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}(x)\,dx\right)^{p-1} \leq \text{const} \tag{4.8}$$

for $w \in \mathcal{W}(\mathbb{R}^N)$ where $Q \subset \mathbb{R}^N$ are arbitrary cubes with edges parallel to the coordinate axes and $|Q|$ is the volume of Q . Formula (4.8) indicates a close connection between the class A_p and the classes $B_p(\mathbb{R}^N)$ and $L^1_{loc}(\mathbb{R}^N)$ essentially used in the foregoing considerations. In particular, we have

$$A_p \subset L^1_{loc}(\mathbb{R}^N) \cap B_p(\mathbb{R}^N)\ .$$

4.10. REMARK. In this paper, we have considered for simplicity the case of spaces of order *one* only. Nevertheless, our considerations can be extended to the space $W^{k,p}(\Omega;S)$ with $k > 1$ mentioned in Introduction. If we introduce the set $\mathcal{M}$ of multi-indices containing some α with $|\alpha| = k$, the weight $S = \{w_\alpha \in \mathcal{W}(\Omega),\ \alpha \in \mathcal{M}\}$ and define $W^{k,p}(\Omega;S)$ as the set of all functions $u = u(x)$ such that $D^\alpha u \in L^p(\Omega;w_\alpha)$ for $\alpha \in \mathcal{M}$, then analogous assertions as in the case $k = 1$ can be obtained at least for certain *special* sets $\mathcal{M}$. For example, $\mathcal{M}$ should have the following structure: If we denote $\mathcal{M}_i = \{\alpha \in \mathbb{N}_0^N, |\alpha| = i\}$, then

(i) $\mathcal{M} \cap \mathcal{M}_k \neq \emptyset$;

(ii) $\theta \in \mathcal{M}$;

(iii) if $\alpha \in \mathcal{M} \cap \mathcal{M}_i$, $1 \leq |i| \leq k$, then there exists at least one $\beta \in \mathcal{M} \cap \mathcal{M}_{i-1}$ such that

$$\alpha - \beta \in \mathcal{M}_1 .$$

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