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ON ANISOTROPIC IMBEDDINGS

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Dedicated to the memory of Svatopluk FUČÍK

Abstract: We consider anisotropic Sobolev and Besov spaces and prove imbeddings theorems of Sobolev and Trudinger type.

Key words: anisotropic Sobolev and Besov spaces, L_p-mixed norm spaces, Lorentz mixed norm spaces, Orlicz spaces.

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0. **Introduction**. Let $I = (0,1)\times(0,1) \subset R^2$, and $\bar{p} = (p_0,p_1,p_2)$ with $1 \leq p_i \leq \infty$, $i = 0,1,2$. We set

$$W^1_{\bar{p}} = W^1_{\bar{p}}(I) = \{f \in L_{p_0}(I) \; ; \; D_i f \in L_{p_i}(I) \; , \; i = 1,2\},$$

and, for $0 < \theta < 1$, $1 \leq s \leq \infty$, $1 \leq p_i, q_i \leq \infty$, $1/r_i = (1-\theta)/p_i + \theta/q_i$, $i = 0,1,2,$

$$B^1_{\bar{r},s} = B^1_{\bar{r},s}(I) = (W^1_{\bar{p}}(I), W^1_{\bar{q}}(I))_{\theta,s;K}$$

(interpolation by the K-method (see, e.g. [3],[9])). The spaces $W^1_{\bar{p}}$ and $B^1_{\bar{r},s}$ are called anisotropic Sobolev and Besov spaces, resp. (of first order).

Let us fix another notation. By $L_{\bar{q},mix} = L_{\bar{q},mix}(I)$, $\bar{q} = (q_1,q_2)$, $1 \leq q_i \leq \infty$, we denote the usual mixed norm space, and, for $0 < \theta < 1$, $1 \leq s \leq \infty$, $1 \leq p_i,q_i \leq \infty$, $1/r_i = (1-\theta)/p_i + \theta/q_i$, $i = 1,2$,

$$L_{\bar{r},s,mix} = L_{\bar{r},s,mix}(I) = (L_{\bar{p},mix}(I), L_{\bar{q},mix}(I))_{\theta,s;K}$$

will be the Lorentz mixed norm space.

The plan of the paper is as follows. First, we obtain imbeddings of $W^1_{\bar{p}}$ and $B^1_{\bar{p},s}$ into Lebesgue and Lorentz mixed norm spaces, resp. when

$$1/p_1 + 1/p_2 > 1 ,$$

and then the limit case

$$1/p_1 + 1/p_2 = 1$$

will be considered.

Imbeddings of type $W^1_{\bar{p}} \hookrightarrow L_q$, $1/p_1 + 1/p_2 > 1$, were the subject of several papers (see, e.g. [8] under some additional assumptions upon the indeces p_i). As for the latter case let us notice that the classical Sobolev theorem states that $W^1_2 \hookrightarrow L_q$ for each $q \in \langle 2, \infty)$ and known counterexamples (see, e.g. [1]) show that there is n o imbedding of type $W^1_2 \hookrightarrow L_\infty$. It is the fundamental paper by Trudinger [11] where a better target space was found, namely, an Orlicz space generated by a Young function asymptotically equivalent to the function $t \mapsto \exp t^2$ at ∞ . Other extensions and refinements were given by Hedberg [5], Moser [6], Strichartz [9], ... An interpolation approach was used in [7] by Peetre and anisotropic spaces (mixed norm anisotropy) were considered by Besov, Il´jin, and Nikolskii in [4]. Let us yet notice that higher order spaces on domains can be taken into consideration, as well, however, it is technically more complicated, in particular, in the case of the Sobolev type imbeddings.

1. The case $1/p_1 + 1/p_2 > 1$.

We suppose that the reader is familiar with elements of the interpolation theory. Basic concepts used here can be found e.g. in [3],[10]. For brevity, we shall frequently use the following standard terminology: If X , X_i , $i = 1,2$, $X \subset X_1 + X_2$, are normed linear spaces and

$$\|x\|_X \leq c \|x\|_{X_1}^{1-\theta} \|x\|_{X_2}^{\theta}$$

for some $\theta \in (0,1)$, we shall say that X is of the class $J(\theta) = J(\theta; X_1, X_2)$.

In the sequel, we shall make use of the following interpolation result by Benedek and Panzone.

1.1. Theorem ([2]). Let $\bar{p}_{(j)} = (p_{(j)1}, \dots, p_{(j)n})$, $j = 1,2$, $1 \leq p_{(j)i} \leq \infty$, $i = 1,\dots,n$, $j = 1,2$, and $0 < \theta < 1$. Let $C = (0,1)^n \subset R^n$ and set $\bar{p} = (p_1, \dots, p_n)$ where

$$1/p_i = (1-\theta)/p_{(1)i} + \theta/p_{(2)i} , \quad i = 1,\dots,n .$$

Then $L_{\bar{p},mix}(C)$ is of the class $J(\theta; L_{\bar{p}(1),mix}(C), L_{\bar{p}(2),mix}(C))$.

1.2. Lemma. Let $1 \leq p_i < \infty$, $i = 0,1,2$, $1/p_1 + 1/p_2 > 1$, $p_0 \geq \max(p_1, p_2)$, and $\alpha > 1$. Then

(i) $L_{(\infty,\alpha),mix}$ is of the class

$$J(1/\alpha ; L_{p_1(\alpha-1)/(p_1-1)} , W_{\bar{p}}^1) ,$$

(ii) $L_{(\alpha,\infty),mix}$ is of the class

$$J(1/\alpha ; L_{p_2(\alpha-1)/(p_2-1)} , W_{\bar{p}}^1) .$$

P r o o f . Using the Beppo-Levi definition of Sobolev (isotropic) Sobolev spaces and the Hölder inequality, we get, for $f \in C^{\infty}(\bar{I})$,

$$\int_0^1 \sup_{0 \le x_2 \le 1} |f(x_1,x_2)|^{\alpha} \, dx_1 \le$$

$$\le c \left(\int_0^1 \int_0^1 |f(x)|^{(\alpha-1)p_2/(p_2-1)} dx \right)^{(p_2-1)/p_2} \cdot \|f\|_{L_{p_2}} +$$

$$+ c \left(\int_0^1 \int_0^1 |f(x)|^{(\alpha-1)p_2/(p_2-1)} dx \right)^{(p_2-1)/p_2} \cdot \|D_2 f\|_{L_{p_2}} =$$

$$= c \|f\|^{\alpha-1}_{L_{(\alpha-1)p_2/(p_2-1)}} \|f\|_{W^1_{\bar{p}}} .$$

Similarly,

$$\int_0^1 \sup_{0 \le x_1 \le 1} |f(x_1,x_2)|^{\alpha} \, dx_2 \le c \|f\|^{\alpha-1}_{L_{(\alpha-1)p_1/(p_1-1)}} \|f\|_{W^1_{\bar{p}}} .$$

1.3. Corollary. Let $1 \le p_i < \infty$, $i = 0,1,2$, and

$$1/p_1 + 1/p_2 > 1 .$$

Then $W^1_{\bar{p}} \hookrightarrow L_{(q_1,q_2),mix}$ where

$$1/q_1 = \theta(p_1p_2 - p_1 + p_2)/(p_1 + p_2 - p_1p_2) ,$$

$$1/q_2 = (1 - \theta)(p_1p_2 + p_1 - p_2)/(p_1 + p_2 - p_1p_2)$$

with $0 < \theta < 1$.

P r o o f . If we set

$$\alpha_1 = (p_1p_2 + p_1 - p_2)/(p_1 + p_2 - p_1p_2) ,$$

$$\alpha_2 = (p_1p_2 - p_1 + p_2)/(p_1 + p_2 - p_1p_2) ,$$

and choose θ so that

$$q = q_1 = q_2 = 2p_1p_2/(p_1 + p_2 - p_1p_2)$$

we get from the preceding lemma

$$\text{(2.1)} \quad \begin{aligned} \|f\|_{L_{(\infty,\ \alpha_1),mix}} &\leq c\ \|f\|_{L_q}^{1 - 1/\alpha_1}\ \|f\|_{W_{\bar{p}}^1}^{1/\alpha_1}\ , \\ \|f\|_{L_{(\alpha_2,\infty),mix}} &\leq c\ \|f\|_{L_q}^{1 - 1/\alpha_2}\ \|f\|_{W_{\bar{p}}^1}^{1/\alpha_2}\ . \end{aligned}$$

According to the Benedek-Panzone theorem, interpolating with $\theta = (p_1p_2 + p_1 - p_2)/2p_1p_2$ we have

$$\text{(2.2)} \qquad \|f\|_{L_q} \leq c\ \|f\|_{W_{\bar{p}}^1}\ .$$

By density argument, (2.2) holds for each function from $W_{\bar{p}}^1$.

Now, let us return to (2.1). With α_1 , α_2 chosen as above (2.1) yields

$$\|f\|_{L_{(\infty,\ \alpha_1),mix}} \leq c\ \|f\|_{W_{\bar{p}}^1}\ ,$$

$$\|f\|_{L_{(\alpha_2,\infty),mix}} \leq c\ \|f\|_{W_{\bar{p}}^1}$$

so that the estimate

$$\|f\|_{L_{(q_1,q_2),mix}} = c\ \|f\|_{W_{\bar{p}}^1}$$

follows by interpolation with $\theta \in (0,1)$ and $1/q_1 = \theta/\alpha_2$, $1/q_2 = (1 - \theta)/\alpha_1$.

If $p_0 < \max(p_1,p_2)$ we can iterate imbeddings with changed p_1,p_2 in order to get the desired target space. Indeed, if

$q^{(n)}$ denotes the exponent in the n-th step it is

$$q^{(n)} = 2p_1^{(n-1)}p_2^{(n-1)}/(p_1^{(n-1)} + p_2^{(n-1)} - p_1^{(n-1)}p_2^{(n-1)}) \geqq$$

$$= 2q^{(n-1)}/(1 + q^{(n-1)} - q^{(n-1)}) = 2q^{(n-1)} ,$$

and $q^{(1)} \geqq 2$ as $W_{\bar{p}}^1 \hookrightarrow W_1^1 \hookrightarrow L_2$.

An interpolation argument applied to the preceding corollary directly gives

1.5. Corollary. Let $1 < p_i < \infty$, $i = 0,1,2$, $1 \leqq s \leqq \infty$, q_j as in Corollary 1.4, $j = 1,2$. Then

$$B_{\bar{p},s}^1 \hookrightarrow L_{(q_1,q_2),s,mix} .$$

Another suitable interpolation (cf. e.g. [3, Thm. 3.8.1]) leads to

1.6. Corollary. Let p_i and q_j , $i = 0,1,2$, $j = 1,2$, be as above, and $s_j < q_j$, $j = 1,2$. Then the imbedding of $W_{\bar{p}}^1$ into $L_{(s_1,s_2),mix}$ and the imbedding of $B_{\bar{p},s}^1$ into $L_{(s_1,s_2),s,mix}$, $1 \leqq s \leqq \infty$ are compact.

2. The case $1/p_1 + 1/p_2 = 1$.

We shall present the basic estimate from which the imbedding into an Orlicz space L_Φ with $\Phi(t) \sim \exp t^2 - 1$ follows easily when considering the Taylor expansion of Φ .

2.1. Theorem. Let $1 \leqq p_0, p_1, p_2 < \infty$, and, say, $p_1 \leqq p_2$. Let $1/p_1 + 1/p_2 = 1$ and $\varepsilon > 0$ be such that $p_2 = 2(1+\varepsilon)^2/(1+2\varepsilon)$. Let $G \subset R^2$ be a bounded domain having the extension property with respect to the anisotropic Besov spaces. Then, for each $r \in (1,\infty)$ there exists a constant $c > 0$ such that

$$\|f\|_{L_q(G)} \leqq cq \, \|f\|_{B^1_{\bar{p},r}}$$

for each $q \geqq 2(1+\varepsilon)$.

P r o o f . Let f be differentiable and suported in a cube $\bar{I} \supset G$. As L_q is of the class $J(1 - 2(1+\varepsilon)/q\,;\, L_{2(1+\varepsilon)}, L_\infty)$ and $W^1_{2(1+\varepsilon)}$ is imbedded into L_∞ we get

$$\|f\|_{L_q} \leqq c\, \|f\|_{L_{2(1+\varepsilon)}}^{1 - 1/(1+\varepsilon) + 2/q} \, \|f\|_{W^1_{2(1+\varepsilon)}}^{1/(1+\varepsilon) - 2/q}$$

therefore, in accordance with the imbedding $W^1_{(1+\varepsilon,1,1+\varepsilon)} \hookrightarrow \hookrightarrow L_{2(1+\varepsilon)}$, it follows that L_q is of the class $J(1/(1+\varepsilon) - 2/q\,;\, W^1_{(1+\varepsilon,1,1+\varepsilon)}, W^1_{2(1+\varepsilon)})$. Set $\eta = 1/(1+\varepsilon)$; then

$$1/p_1 = 1 - \eta + \eta/2(1+\varepsilon),$$
$$1/p_2 = (1-\eta)/(1+\varepsilon) + \eta/2(1+\varepsilon).$$

Let us interpolate between $W^1_{(1+\varepsilon,1,1+\varepsilon)}$ and $W^1_{2(1+\varepsilon)}$ with parameters η and $r \in (1,\infty)$. As $W^1_{2(1+\varepsilon)} \hookrightarrow W^1_{(1+\varepsilon,1,1+\varepsilon)}$ it is known from the interpolation theory that there exists a representation $g_f : (0,1) \longrightarrow W^1_{2(1+\varepsilon)}$ such that

$$f = \int_0^1 g_f(t)\, dt/t \quad , \text{ and}$$

$$J(t, g_f(t); W^1_{(1+\varepsilon,1,1+\varepsilon)}, W^1_{2(1+\varepsilon)}) \leqq$$

$$\leqq c \quad K(t, f; W^1_{(1+\varepsilon,1,1+\varepsilon)}, W^1_{2(1+\varepsilon)}) .$$

Set $\mu = 1/(1+\varepsilon) - 2/q$. Integrating and using the Hölder inequality,

$$\|f\|_{L_q} \leq c \int_0^1 t^{-\mu} K(t,f;W^1_{(1+\varepsilon,1,1+\varepsilon)}, W^1_{2(1+\varepsilon)}) \, dt/t \leq$$

$$\leq c \left(\int_0^1 t^{(\eta-\mu)r/(r-1)-1} dt \right)^{(r-1)/r} \|f\|_{B^1_{\bar{p},r}} =$$

$$= cq \|f\|_{B^1_{\bar{p},r}} .$$

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