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BOUNDARY VALUE PROBLEMS WITH NONLINEARITIES HAVING INFINITE JUMPS

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Dedicated to the memory of Svatopluk FUČÍK

Abstract: We extend some results of Ward for nonlinear perturbations of linear operators whose kernel is made of constant functions to the case where the kernel is spanned by a positive function. Applications are given which extend earlier results of Aguinaldo-Schmitt and Castro.

Key words: Boundary value problems for ordinary differential equations, jumping nonlinearities, Leray-Schauder method.

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1. Introduction. In his fundamental work on nonlinear noncoercive equations, Fučík has emphasized the important concept of "jumping nonlinearity" and has given in [6] the first systematic study of the Dirichlet problem for second order ordinary differential equations with jumping nonlinearities, namely

$$x''(t) + g(x(t)) = h(t),$$

$$x(0) = x(\pi) = 0$$

with $\lim_{x \to -\infty} g(x)/x \neq \lim_{x \to +\infty} g(x)/x$. As most of Fučík's papers, [6] not only contains significant results but also a number of interesting open questions. One of them was solved by Aguinaldo and Schmitt [1] who proved that the problem

(1.1) $$x''(t) + x(t) - \alpha x^{-}(t) = h(t)$$
$$x(0) = x(\pi) = 0$$

with $\alpha > 0$, $x^{-} = \max(-x,0)$ and h continuous has a solution if and only if

(1.2) $$\int_0^{\pi} h(s) \sin s \, ds \neq 0.$$

Aguinaldo and Schmitt use a continuation theorem due to the author (see e.g. [8]) and obtain the required a priori bounds by a delicate argument linked to the special nature of the nonlinear term in (1.1). Their result was generalized by Castro [5] who proved the sufficient condition (1.2) for the more general problem

(1.3) $$x''(t) + x(t) + g(x(t)) = h(t),$$
$$x(0) = x(\pi) = 0$$

with $g:R \longrightarrow R$ continuous, $g(x) = 0$ for $x \geq 0$ and $g(x)/x \longrightarrow \alpha > 0$ when $x \longrightarrow -\infty$. Castro's proof uses a rather sophisticated variational argument which strongly uses the sublinear character of g.

The aim of this paper is to provide a partial extension of the method initiated by Ward [9] for the study of periodic solutions of semi-linear ordinary differential equations whose linear part only admits constant periodic solutions. This extension allows the kernel of the linear part to be spanned by a positive function and provides generalizations of the results of Aguinaldo-Schmitt and Castro to ordinary differential equations of arbitrary order and to some classes of nonlinearities which do not have necessarily a linear growth. Finally, the underlying abstract tool is simply a continuation theorem of Leray-Schauder type [8] and the corresponding a priori bounds are obtained in a rather simple way.

In the case of (1.3), our theorem implies the existence of a solution when (1.2) holds when $h \in L^1(0,\pi)$, $g(x) = 0$ for $x \geq 0$ and

$$\lim_{x \to -\infty} \sup g(x) = -\infty .$$

Another easy consequence of our results is that the problem, with $h \in L^1(0,\pi)$

$$x''(t) + x(t) + \alpha \exp x(t) = h(t)$$

$$x(0) = x(\pi) = 0$$

with $\alpha > 0$ has a solution if and only if

$$\int_0^{\pi} h(t) \sin t \, dt > 0. \tag{1.4}$$

Finally, our method easily shows that (1.4) is also sufficient for the existence of one solution for the problem

$$x'''(t) + x'(t) + \alpha \exp [x(t) + \sin x'(t)] = h(t),$$

$$x(0) = x'(0) = x'(\pi) = 0.$$

2. <u>Preliminary results on linear operators.</u> Let $I = [a,b]$, $k \geq 0$ an integer, $C^k(I)$ the Banach space of real functions of class C^k on I. with the usual norm $|u|_{C^k} = \sum_{j=0}^{k} \max_{t \in I} |u^{(j)}(t)|$, $L^1(I)$ the Banach space of real functions L-integrable on I with the usual norm

$$|u|_{L^1} = \int_I |u(t)| \, dt.$$

Let $L: D(L) \subset C^k(I) \longrightarrow L^1(I)$ be a closed linear operator having the following properties.

(L_1) $\ker L = \operatorname{span}\{\varphi\}$, with $\varphi \in D(L)$ such that $\varphi(t) > 0$ for a.e. $t \in I$ and $\int_I \varphi(t)dt = 1$.

(L_2) $\operatorname{Im} L = \{y \in L^1(I): \int_I y(t) \psi(t)dt = 0\}$ for some $\psi \in L^{\infty}(I)$ such that $\int_I \varphi(t) \psi(t)dt = 1$ and $\psi(t) > 0$ for a.e. $t \in I$.

Let us denote by $L^1_\psi(I)$ the Banach space of real functions L-integrable on I with the norm

$$|u|_\psi = \int_I |u(t)|\, \psi\,(t)dt,$$

and let us introduce the following closed subspaces of $C^k(I)$, $L^1(I)$, $L^1_\psi(I)$,

$\tilde{C}^k(I)$ is a topological direct summand of ker L in $C^k(I)$,

$$\tilde{L}^1(I) = \{y \in L^1(I):\ \int_I y(t)\, \psi\,(t)dt = 0\},$$

$$\tilde{L}^1_\psi(I) = \{y \in L^1_\psi(I):\ \int_I y(t)\, \psi\,(t)dt = 0\}.$$

We introduce another assumption upon L.

(L_3) There exists a continuous linear operator $A: C^k(I) \longrightarrow$ $\longrightarrow L^1(I)$ such that $L - A: D(L) \subset C^k(I) \longrightarrow L^1(I)$ is one-to-one and onto and such that for some $M \geq 0$ and all $y \in L^1_\psi(I)$, one has

$$|(L - A)^{-1} y|_{C^k} \leq M |y|_\psi .$$

Proposition 1. If conditions (L_1) to (L_3) hold, there exists $\Lambda \geq 0$ such that, for each $x = \bar{x} + \tilde{x} \in D(L)$, with $\bar{x} \in \ker L$, $\tilde{x} \in \tilde{C}^k(I)$, one has

$$|\tilde{x}|_{C^k} \leq \Lambda\, |L\tilde{x}|_\psi = \Lambda\, |Lx|_\psi .$$

Proof. The restriction of L to $D(L) \cap \tilde{C}^k(I)$ being one-to-one and onto $\tilde{L}^1_\psi(I)$, it suffices, by the closed graph theorem, to show that this restriction is a closed operator. By condition (L_3), $(L - A)^{-1}: L^1_\psi(I) \longrightarrow C^k(I)$ is continuous and hence $L - A: D(L) \subset C^k(I) \longrightarrow L^1_\psi(I)$ is closed. Let $(\tilde{x}_n)$ be a sequence in $D(L) \cap \tilde{C}^k(I)$ such that $\tilde{x}_n \longrightarrow \tilde{x} \in \tilde{C}^k(I)$ and $L\tilde{x}_n \longrightarrow \tilde{y} \in \tilde{L}^1_\psi(I)$ in $L^1_\psi(I)$. Then $A\tilde{x}_n \longrightarrow A\tilde{x}$ in $L^1(I)$ and hence in $L^1_\psi(I)$ so that $(L - A)\tilde{x}_n \longrightarrow \tilde{y} - A\tilde{x}$ in $L^1_\psi(I)$. By the closedness of $L - A$ as a mapping between $D(L) \subset C^k(I)$ and $L^1_\psi(I)$, we have $\tilde{x} \in D(L)$ and

$\tilde{y} - A\tilde{x} = L\tilde{x} - A\tilde{x}$, so that $\tilde{y} = L\tilde{x}$. Thus $L: D(L) \subset \tilde{C}^k(I) \longrightarrow \tilde{L}^1_\psi(I)$ is closed, and the proof is complete.

In our applications, L will be a differential operator and condition (L_3) can often be deduced from the following more concrete assumption.

(L_3') $k = 0$ and there exists $A: C^0(I) \longrightarrow L^1(I)$ linear continuous and $G_A \in C^0(I \times I)$ such that $L - A: D(L) \subset C^0(I) \longrightarrow L^1(I)$ is one-to-one and onto,

$$(2.1) \qquad (L - A)^{-1} y(t) = \int_I G_A(t,s) y(s) ds, \quad t \in I$$

and $G_A / \psi \in L^\infty(I \times I)$.

Proposition 2. If conditions (L_1), (L_2) and (L_3') hold, then the conclusion of Proposition 1 is valid with $k = 0$.

Proof. By (2.1), we have, for each $y \in L^1(I)$ and $t \in I$,

$$|(L-A)^{-1} y(t)| = \left| \int_I [G_A(t,s)/\psi(s)]\, y(s) \psi(s) ds \right| \leq$$

$$\leq |G_A/\psi|_{L^\infty} \int_I |y(s)| \psi(s) ds = |G_A/\psi|_{L^\infty} |y|_\psi .$$

Hence condition (L_3) with $k = 0$ holds and the result follows from Proposition 1.

Example 1. As a first example, let L be defined by $D(L) = \{x \in C^0[0,\pi] : x$ is of class C^1 on $I = [0,\pi]$, x' is absolutely continuous on I and $x(0) = x(\pi) = 0\}$, $L: D(L) \subset C^0(I) \longrightarrow L^1(I)$, $x \longmapsto -x'' - x$, so that L is closed, $\ker L = \operatorname{span}\{\sin(\cdot)\}$, $\operatorname{Im} L = \{y \in L^1(I) : \int_I y(t) \sin t\, dt = 0\}$ and we can take $\tilde{C}^0(I) = \{x \in C^0(I) : \int_I x(t) \sin t\, dt = 0\}$.
Moreover, for $A = -\mathrm{Id}$, $G_A = G$, the usual Green function of $-d^2/dt^2$ with the Dirichlet boundary conditions on $[0,\pi]$, namely

$$G(t,s) = \begin{cases} (s/\pi)(\pi - t) \text{ if } 0 \le s \le t \le \pi, \\ (t/\pi)(\pi - s) \text{ if } 0 \le t \le s \le \pi. \end{cases}$$

Therefore, if $0 < s \le t < \pi$, we have $0 < \pi - t \le \pi - s$ and hence

$$0 \le G(t,s)/\sin s \le s(\pi - s)/\pi \sin s \le C$$

as $\lim_{s \to 0} s/\sin s = \lim_{s \to \pi} (\pi - s)/\sin s = 1$, and similarly for $0 < t \le s < \pi$. Thus, all the conditions of Proposition 2 are satisfied and hence

$$|\tilde{x}|_{C^0} \le \Lambda \int_I |x''(t) + x(t)| \sin t \, dt = \Lambda |x'' + x|_{\sin}$$

for all $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} = c \sin(\cdot)$ and $\int_0^\pi \tilde{x}(t) \sin t \, dt = 0$.

Example 2. For a less direct application of the above result, let L_1 be defined by $D(L_1) = \{x \in C^1(I): I = [0,\pi]$, $x \in C^1(I)$, x'' is absolutely continuous on I and $x(0) = x'(0) = x'(\pi) = 0\}$, $L_1: D(L_1) \subset C^1(I) \longrightarrow L^1(I)$, $x \longmapsto -x''' - x'$, so that L_1 is closed,

$$\ker L_1 = \operatorname{span}\{1 - \cos(\cdot)\}$$

$$\operatorname{Im} L_1 = \{y \in L^1(I): \int_I y(t) \sin t \, dt = 0\}.$$

Let $y = x'$, so that, as $x(0) = 0$, $x(t) = \int_0^t y(s)\, ds$, and

$$y(0) = y(\pi) = 0, \quad -x''' - x' = -y'' - y.$$

Therefore, by Example 1 applied to y, we have, for all $y = \bar{y} + \tilde{y} \in D(L)$ with $\bar{y}(t) = c \sin t$ and $\int_0^\pi \tilde{y}(t) \sin t \, dt = 0$,

$$|\tilde{y}|_{C^0} \le \Lambda |\tilde{y}'' + \tilde{y}|_{\sin} = \Lambda |y'' + y|_{\sin} = \Lambda |x''' + x'|_{\sin}.$$

Consequently, as

$$x(t) = y(t) = c \sin t + \tilde{y}(t)$$

we have

$$|x(\cdot) - c \sin(\cdot)|_{C^0} \le \Lambda |x''' + x'|_{\sin}$$

$$|x(\cdot) - c(1 - \cos(\cdot))|_{C^0} = |\int_0^{\cdot} \tilde{y}(s)ds|_{C^0} \leq \pi\Lambda |x''' + x'|_{\sin}$$

for all $x \in D(L_1)$. Thus, if we define the bounded linear operator $P: C^1(I) \longrightarrow C^1(I)$ by

$$(Px)(t) = [(2/\pi) \int_0^{\pi} x'(s) \sin s \, ds] (1 - \cos t)$$

it is easy to check that Im P = ker L_1 and that $P^2 = P$ so that we can take $\tilde{C}^1(I) = \ker P$. Thus, if we write $x(t) = \bar{x}(t) + \tilde{x}(t)$ with $\bar{x} = Px$ and $\tilde{x} = (I - P)(x)$, then, with the notations above, $\bar{x}(t) = c(1 - \cos t)$, and the above inequalities can be written

$$|\tilde{x}'|_{C^0} \leq \Lambda |x''' + x'|_{\sin}, \quad |\tilde{x}|_{C^0} \leq \pi\Lambda |x''' + x'|_{\sin}$$

i.e.

$$|\tilde{x}|_{C^1} \leq (1 + \pi)\Lambda |x''' + x'|_{\sin}.$$

3. <u>An existence theorem for abstract boundary value problems</u>. Let now $f: I \times R^{k+1} \longrightarrow R$ be such that $f(t,\cdot)$ is continuous on R^{k+1} for a.e. $t \in I$ and $f(\cdot,y)$ is measurable on I for each $y \in R^{k+1}$. Assume moreover that for each $r > 0$, there exists $a_r \in L^1(I)$ such that

$$|f(t,y)| \leq a_r(t)$$

whenever $t \in I$ and $|y| \leq r$. Such an f will be called a Carathéodory function for $L^1(I)$.

Let us introduce the following condition of Ward type (see [9] and [2 - 4, 10] for various extensions).

(f_1) There exists $\gamma \in L^1(I)$ and $\varepsilon = \pm 1$ such that $|\gamma|_\gamma > 0$ and

$$|f(t,y)| \leq \varepsilon f(t,y) + \gamma(t)$$

for a.e. $t \in I$ and all $y \in R^{k+1}$.

Such a condition expresses the fact that f is either bounded

below or bounded above with respect to y.

We shall consider the following abstract boundary value problem

(3.1) $\quad (Lx)(t) = f(t,x(t),x'(t),\dots,x^{(k)}(t)), \quad t \in I$

where L is a linear operator of the type considered in section 2 and satisfying moreover the following compactness condition.

(L_4) The inverse of the operator $L:D(L) \subset \tilde{C}^k(I) \longrightarrow \tilde{L}^1(I)$ is compact.

The conditions (L_1) to (L_4) and (f_1) are not sufficient to insure the existence of a solution for (3.1), as shown by the trivial example

$$x''(t) + x(t) = \sin t$$
$$x(0) = x(\pi) = 0$$

for which they are satisfied and which has no solution.

We introduce a supplementary sign condition upon f.

(f_2) There exists $\sigma = \pm 1$ and $\rho > 0$ such that

$$\sigma \int_I f(t, c\varphi(t) + \tilde{v}(t),\dots,c\varphi^{(k)}(t) + \tilde{v}^{(k)}(t))\psi(t)dt \leq 0$$

whenever $c \leq -\rho$ and $|\tilde{v}|_{C^k} \leq \Lambda |\gamma|_\psi$, and

$$\sigma \int_I f(t,c\varphi(t) + \tilde{v}(t),\dots,c\varphi^{(k)}(t) + \tilde{v}^{(k)}(t))\psi(t)dt \geq 0$$

whenever $c \geq \rho$ and $|\tilde{v}|_{C^k} \leq \Lambda |\gamma|_\psi$, where Λ is given by Proposition 1 and $\tilde{v} \in D(L)$ with $\int_I \tilde{v}(t)\varphi(t)\,dt = 0$.

We can now prove the following existence theorem.

Theorem 1. Assume that L satisfies the conditions (L_1) to (L_4) and that f satisfies the conditions (f_1) and (f_2). Then problem (3.1) has at least one solution.

Proof. Let $F:C^k(I) \longrightarrow L^1(I)$ be the Nemitsky operator

associated to f and defined by

$$Fx = f(\cdot, x(\cdot), x'(\cdot), \ldots, x^{(k)}(\cdot)),$$

so that (3.1) is equivalent to the abstract equation

$$Lx = Fx$$

in $C^k(I)$, and F is L-completely continuous on $C^k(I)$. Let $x(t) = \bar{x}(t) + \tilde{x}(t)$, with $\bar{x} \in \ker L$ and $\tilde{x} \in \tilde{C}^k(I)$, and define $G: C^k(I) \to L^1(I)$ by

$$G x = (1 - |\bar{x}|_{C^k})^{-1} (\delta \eta / 2) \bar{x}(\cdot), \quad \eta = |\gamma|_\psi / |\psi|_{L^1},$$

so that G is odd, L-completely continuous and

$$|(Gx)(t)| \leq \eta / 2$$

for a.e. $t \in I$. By Theorem IV.3 and Proposition II.18 of [8], (3.1) will have a solution if the set of possible solutions of the family of equations

$$Lx = (1 - \lambda) Gx + \lambda Fx, \quad \lambda \in [0,1[, \tag{3.2}$$

is a priori bounded independently of λ. Let $\lambda \in [0,1[$ and x be a possible solution of (3.2). Then,

$$0 = (1 - \lambda) \int_I (Gx)(t) \psi(t)\, dt + \lambda \int_I (Fx)(t) \psi(t)\, dt \tag{3.3}$$

and

$$\int_I |(Lx)(t)| \psi(t) dt \leq (1 - \lambda) \int_I |(Gx)(t)| \psi(t) dt + \lambda \int_I |(Fx)(t)| \psi(t) dt.$$

Using condition (f_1) and (3.3), the last inequality implies that

$$|Lx|_\psi \leq (1 - \lambda) |\gamma|_\psi / 2 + \lambda \varepsilon \int_I (Fx)(t) \psi(t(dt) +$$

$$+ \lambda \int_I \gamma(t) \psi(t)\, dt = (1 - \lambda) |\gamma|_\psi / 2 - \varepsilon (1 - \lambda) \int_I (Gx)(t) \psi$$

$$(t)\, dt + \lambda \int_I \gamma(t) \psi(t) dt \leq (1 - \lambda) |\gamma|_\psi + \lambda |\gamma|_\psi = |\gamma|_\psi .$$

Consequently, using Proposition 1, we have

$$|\tilde{x}|_{C^k} \leq \Lambda |\gamma|_\psi . \tag{3.4}$$

If we set $\bar{x}(t) = c \varphi(t)$, then by (3.4) and condition (f_2) we get, if $c \leq -\rho$,

$$(1 - \lambda) \eta \int_I (1 + |\bar{x}|_{C^k})^{-1} \bar{x}(t) \psi(t)\, dt + \sigma \lambda \int_I f(t, x(t), \ldots$$

$$\ldots,x^{(k)}(t))\psi(t)\,dt = (1-\lambda)(1+|\bar{x}|_{C^k})^{-1}\eta\, c +$$

$$+ \delta\lambda\int_I f(t,c\varphi(t)+\tilde{x}(t),\ldots,c\varphi^{(k)}(t)+\tilde{x}^{(k)}(t))\psi(t)\,dt \leq$$

$$\leq (1-\lambda)(1+|\bar{x}|_{C^k})^{-1}\eta\, c \leq -(1-\lambda)(1+|\bar{x}|_{C^k})^{-1}\eta\rho < 0$$

so that (3.3) cannot hold. Similarly if $c \geq \rho$, which implies that we have necessarily

$$|c| < \rho,$$

and hence, by (3.4),

$$|x|_{C^k} \leq |c\varphi|_{C^k} + |\tilde{x}|_{C^k} < \rho|\varphi|_{C^k} + \Lambda|\gamma|_\Psi = R$$

and the proof is complete.

It has already been noticed that the sign condition (f_2) contains as special case Landesman-Lazer conditions of the following type.

(f_2') $k = 0$, there exist functions $\sigma'_+ \in L^1(I)$ and $\sigma'_- \in L^1(I$ such that

$$f(t,y) \geq \sigma'_+(t) \text{ if } y \geq 0$$

and

$$f(t,y) \leq \sigma'_-(t) \text{ if } y \leq 0,$$

and the measurable functions μ_1 and μ_2 defined by

$$\mu_1(t) = \limsup_{y\to-\infty} f(t,y),\quad \mu_2(t) = \liminf_{y\to+\infty} f(t,y),$$

are such that

$$\int_I \mu_1(t)\psi(t)dt < 0 < \int_I \mu_2(t)\psi(t)dt.$$

We give a proof for completeness.

<u>Proposition 3</u>. Condition (f_2') implies condition (f_2) with $k = 0$ and $\sigma' = 1$.

<u>Proof</u>. If it is not the case, there will exist sequences (c_n) and $(\bar{x}_n)$ with $|\bar{x}_n|_{C^0} \leq \Lambda|\gamma|_\Psi$ such that either $c_n \to -\infty$

and

$$\int_I f(t,c_n\varphi(t) + \tilde{v}_n(t))\psi(t)\ dt > 0$$

or such that $c_n \to +\infty$ and

$$\int_I f(t,c_n\varphi(t) + \tilde{v}_n(t))\psi(t)\ dt < 0.$$

Considering, say, the first case, we have, for a.e. $t \in I$,

$$c_n\varphi(t) + \tilde{v}_n(t) \longrightarrow -\infty$$

if $n \longrightarrow \infty$, and hence, by Fatou's lemma.

$$0 \leq \limsup_{n\to\infty} \int_I f(t,c_n\varphi(t) + \tilde{v}_n(t))\psi(t)\ dt \leq$$

$$\leq \int_I \limsup_{n\to\infty} f(t,c_n\varphi(t) + \tilde{v}_n(t))\psi(t)\ dt \leq$$

$$\leq \int_I \limsup_{x\to-\infty} f(t,x)\psi(t)\ dt = \int_I \mu_1(t)\psi(t)\ dt < 0,$$

a contradiction.

One can show similarly that the following condition (f_2^n) implies condition (f_2) with $k = 0$ and $\sigma = -1$.

(f_2^n) $k = 0$, there exist functions $\sigma_+ \in L^1(I)$ and $\sigma_- \in L^1(I)$ such that

$$f(t,y) \leq \sigma_+(t) \text{ if } y \geq 0$$

$$f(t,y) \geq \sigma_-(t) \text{ if } y \leq 0$$

and the measurable functions μ_1 and μ_2 defined by

$$\mu_2(t) = \liminf_{x\to-\infty} f(t,x), \quad \mu_1(t) = \limsup_{x\to+\infty} f(t,x)$$

are such that

$$\int_I \mu_1(t)\psi(t)\ dt < 0 < \int_I \mu_2(t)\psi(t)\ dt.$$

Let us mention the following obvious Corollary of Theorem 1.

Corollary 1. Assume that L satisfies conditions (L_1),(L_2), (L_3'),(L_4) and that f satisfies condition (f_1) and (f_2') or (f_2^n). Then the problem

$$(Lx)(t) = f(t,x(t))$$

has at least one solution.

In particular, if $f(t,x) = h(t) - g(x)$, with $h \in L^1(I)$, $g: R \longrightarrow R$ is continuous, $g(x) = 0$ for $x \geq 0$, $\limsup_{x \to -\infty} g(x) = -\infty$, then condition (f_1) holds with $\varepsilon = 1$ and

$$\liminf_{x \to -\infty} f(t,x) = h(t) - \limsup_{x \to -\infty} g(x) = +\infty,$$

$$\limsup_{x \to +\infty} f(t,x) = h(t) - \liminf_{x \to +\infty} g(x) = h(t),$$

so that condition (f_2^n) becomes here

$$\int_I h(t)\psi(t)\,dt < 0.$$

In the special case where $Lx = x'' + x$ with the Dirichlet boundary conditions on $[0,\pi]$, $\psi(t) = \sin t$, all conditions (L_1), (L_2), (L_3'), (L_4) are satisfied (see Example 1 in Section 2) and we obtain the generalization of the results of Aguinaldo-Schmitt and Castro announced in the Introduction.

If $f(t,x) = h(t) - \alpha \exp x$, with $h \in L^1(I)$ and $\alpha > 0$, then condition (f_1) holds with $\varepsilon = -1$ and

$$\liminf_{x \to -\infty} f(t,x) = h(t), \quad \limsup_{x \to +\infty} f(t,x) = -\infty,$$

so that condition (f_2^n) becomes

$$0 < \int_I h(t)\psi(t)\,dt.$$

In the special case where $Lx = x'' + x$ with the Dirichlet boundary conditions on $[0,\pi]$, we again find the condition

$$0 < \int_I h(t) \sin t\,dt$$

announced in the Introduction. Notice that when $\alpha < 0$, our result can also be applied and furnishes the existence condition

$$\int_I h(t) \sin t\,dt < 0,$$

but, in contrast to the case where $\alpha > 0$, the situation with $\alpha < 0$ can also be treated by the method of upper and lower

solutions (see e.g. [7], Chapter 32). Notice that those conditions are also necessary for the existence of a solution.

As a last example, let us consider the following boundary value problem

$$x''(t) + x'(t) + \alpha \exp[x(t) + \sin x'(t)] = h(t) \tag{3.5}$$
$$x(0) = x'(0) = x'(\pi) = 0$$

where $h \in L^1(I)$, $I = [0, \pi]$ and $\alpha \neq 0$. It follows easily from Example 2 that a necessary condition for the existence of a solution of (3.5) is that

$$\alpha \int_0^{\pi} h(t) \sin t \, dt > 0. \tag{3.6}$$

Combining the results of Example 2 with Theorem 1, it is easy to show that this condition is also sufficient.

Remark 1. In the case of nonlinear perturbations of linear operators whose kernel is made of constant functions, Ward's growth conditions on the nonlinear term f are of the form

$$|f(t,y)| \leq \varepsilon f(t,y) + \beta |y| + \gamma(t)$$

with β sufficiently small. Our approach in the setting of a kernel spanned by a positive function does not seem to extend easily to such a growth condition with $\beta > 0$ and it is an open problem to know if the results of this paper are true or not in this more general setting.

Remark 2. The same method can obviously be applied to boundary value problems for functional-differential equations, as well as to boundary value problems for systems of equations, with generalized Ward conditions in the line of [2, 3].

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