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REMOVABLE SINGULARITIES OF SOLUTIONS OF THE HEAT EQUATION WITH SPECIAL GROWTH

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Let us denote by ρ the metric on $\mathbb{R}_{m+1}$ defined for any $x = (x_1,\dots,x_{m+1})$, $y = (y_1,\dots,y_{m+1}) \in \mathbb{R}_{m+1}$ by the formula

$$\rho(x,y) = (|x_{m+1} - y_{m+1}| + \sum_{i=1}^{m} |x_i - y_i|^2)^{1/2}.$$

For any $q \geq 0$ we shall define set functions $\mathcal{M}^q$ and $\mathcal{H}^q$ as follows. If A is a Borel set in $\mathbb{R}_{m+1}$ then

$$\mathcal{M}^q(A) = \limsup_{\varepsilon \to 0_+} \lambda(\{x \in \mathbb{R}_{m+1} | \operatorname{dist}_\rho(x,A) \leq \varepsilon\})/\varepsilon^{m+2-q}$$

and

$$\mathcal{H}^q(A) = \sup_{\varepsilon > 0} \inf \{\sum_{i=1}^{\infty} (\operatorname{diam}_\rho S_i)^q | A \subset \bigcup_{i=1}^{\infty} S_i \ \&$$
$$\& \ (\forall i = 1,2,\dots : \operatorname{diam}_\rho S_i \leq \varepsilon)\}$$

where λ denotes the Lebesgue measure in $\mathbb{R}_{m+1}$. For metric ρ with respect to the heat equation compare [3].

__Theorem 1__: Let G be an open set in $\mathbb{R}_{m+1}$ and F be a relatively closed set in G. Let $0 \leq q \leq m$ and suppose f is a locally integrable function in G satisfying

$$f(x) = o(\operatorname{dist}_\rho(x,F)^{-q}) \quad (\text{resp. } f(x) = O(\operatorname{dist}_\rho(x,F)^{-q}))$$

as $\operatorname{dist}_\rho(x,F) \to 0_+$ locally in G. If f satisfies (in the sense of distributions) the heat equation $(\partial/\partial x_{m+1} - \sum_{i=1}^{m} \partial^2/\partial x_i^2) f = 0$ on $G \setminus F$ and $\mathcal{M}^{m-q}(K) < +\infty$ (resp. $\mathcal{M}^{m-q}(K) = 0$) for any compact set $K \subset F$ then f satisfies the same equation on G.

__Theorem 2__: Let K be a compact set in $\mathbb{R}_{m+1}$ and let $0 < q \leq m$. Suppose $\mathcal{H}^{m-q}$ is not σ-finite on K (resp. $\mathcal{H}^{m-q}(K) > 0$). Then there exists a locally integrable function f on $\mathbb{R}_{m+1}$ satisfying

$$f(x) = o(\operatorname{dist}_\rho(x,K)^{-q}) \quad (\text{resp. } f(x) = O(\operatorname{dist}_\rho(x,K)^{-q}))$$

as $\operatorname{dist}_\rho(x,K) \to 0_+$ such that f is a solution of the heat equation on $\mathbb{R}_{m+1} \setminus K$ but not on $\mathbb{R}_{m+1}$. Such a function f can be found as a heat potential of some non-negative Radon measure supported by K.

The proofs of both Theorem 1 and Theorem 2 are included in my thesis submitted to the Faculty of Mathematics and Physics of the Charles University in April 1984. For Theorem 1 compare the Bochner's removable singularity theorem as formulated in [2]. Note that our Theorem 1 is not implied by the Bochner's theorem. For Theorem 2 compare an analogous result of Hamann in [1] dealing with elliptic equations.

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A CLOSED SEPARABLE SUBSPACE NOT BEING A RETRACT OF $\beta\mathbb{N}$

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D. Maharam [M] proved that the following are equivalent:

(a) For each ideal $I \subseteq \mathcal{P}(\mathbb{N})$, if there is a one-to-one homomorphism from $\mathcal{P}(\mathbb{N})/I$ to $\mathcal{P}(\mathbb{N})$, then there is a lifting from $\mathcal{P}(\mathbb{N})/I$ to $\mathcal{P}(\mathbb{N})$, too;

(b) every non-void closed separable subspace of $\beta\mathbb{N}$ is a retract of $\beta\mathbb{N}$,

and has raised the question, whether (a) or (b) is a true statement.

The answer to the Maharam's problem is in negative. We can prove the two theorems below.

Theorem 1. There exists a subspace $X \subseteq \beta\mathbb{N} - \mathbb{N}$ satisfying the following:

(1) $X = \bigcup_{n\in\omega} X_n$, where $|X_0| = 1$ and for each $n \in \omega$, the set X_n is countable discrete;

(2) for each $n < m < \omega$, $X_n \subseteq \overline{X}_m - X_m$;

(3) for each $n < \omega$ and for each $x \in X_n$, x is a $\mathfrak{c}$ - OK point in $\overline{X}_{n+1} - X_{n+1}$;

(4) suppose $\{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbb{N})$ to be a family of sets such that for some $n_0 < \omega$, $U_0^* \cap X_{n_0}$ is finite and for each $i < k < \omega$, $U_i^* \cap X_{n_0+i} \subseteq U_k^*$. Then there is a family $\{V_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\mathbb{N})$ such that for each $\alpha \in \mathfrak{c}$, $V_\alpha^* \supseteq X \cap \bigcap_{k\in\omega} U_k^*$ and for each $k < \omega$ and for each finite set $\alpha_0 < \alpha_1 < \ldots < \alpha_k < \mathfrak{c}$, $\bigcap_{i\le k} V^*_{\alpha_i} \subseteq \bigcap_{i\le k} U_i^*$;

(5) for each mapping $f: \mathbb{N} \to X$ there is a set $T \subseteq \mathbb{N}$ and an integer $n_1 < \omega$ such that $T^* \cap X \neq \emptyset$ and for each $n > n_1$, $X_n \cap \overline{f[T]} \cap X_{n+1} = \emptyset$.

Theorem 2. If a subspace $X \subseteq \beta\mathbb{N}$ satisfies (1) - (5) from Theorem 1, then $\overline{X}$ is not a retract of $\beta\mathbb{N}$.

It should be noted that the first example of a closed separable subspace of $\beta\mathbb{N}$ which is not a retract of $\beta\mathbb{N}$ was given by M. Talagrand under CH in [T] and the second one by A. Szymański under MA in [S].

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[T] M. Talagrand: Non existence de relèvement pour certaines mesures finiement additives et retractés de βN, Math. Ann. 256(1981), 63-66.

SHORT BRANCHES IN RUDIN-FROLÍK ORDER

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Rudin-Frolík order of types of ultrafilters in βN has the following properties:

(1) each type of ultrafilters has at most $2^{\aleph_0}$ predecessors, [2],

(2) the cardinality of each branch is at least $2^{\aleph_0}$.

Thus, in Rudin-Frolík order the cardinality of branches can be only $2^{\aleph_0}$ or $(2^{\aleph_0})^+$. It was shown in [1] that there exists a chain order - isomorphic to $(2^{\aleph_0})^+$. Hence, the existence of a branch of cardinality $(2^{\aleph_0})^+$ is proved.

The following result solves the problem of the existence of a branch having smaller cardinality.

Theorem. In Rudin-Frolík order there exists an unbounded chain order-isomorphic to ω_1.

By the properties (1) and (2) the branch containing this chain has cardinality $2^{\aleph_0}$.

References: [1] E. Butkovičová: Long chains in Rudin-Frolík order, Comment. Math. Univ. Carolinae 24(1983), 563-570.
[2] Z. Frolík: Sums of ultrafilters, Bull. Amer. Math. Soc. 73(1967), 87-91.

RESULTS ON DISJOINT COVERING SYSTEMS ON THE RING OF INTEGERS

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A system of congruence classes

(1) $a_1 (\mathrm{mod}\ n_1),\ a_2 (\mathrm{mod}\ n_2),\ \ldots,\ a_k (\mathrm{mod}\ n_k)$

will be called a disjoint covering system (DCS) if for every integer x there is exactly one $i \in \{1, 2, \ldots, k\}$ such that $x \equiv a_i (\mathrm{mod}\ n_i)$. The integers $n_1, n_2, \ldots, n_k$ will be called moduli of (1) and their least common multiple will be called the common modulus of (1).

If $k > 1$ then no two moduli of (1) are relatively prime. This condition can be expressed in the form

(2) $$\bigwedge_{i=1}^{k} \bigwedge_{j=1}^{k} \varphi(n_i, n_j)$$

where $\varphi(x, y)$ is the formula

$$\exists z \exists u \exists v (z \neq 1 \wedge z.u = x \wedge z.v = y)$$

Consider more generally the formulae of the form

$$(3) \qquad \bigwedge_{i_1=1}^{k} \bigwedge_{i_2=1}^{k} \cdots \bigwedge_{i_r=1}^{k} \psi(n_{i_1}, n_{i_2}, \ldots, n_{i_r})$$

which are true for all DCS (1) with $k > 1$, where $\psi(x_1, \ldots, x_r)$ is a first-order formula with the only non-logical symbol "." for multiplying. The main result of [1] is that every such formula (3) is a consequence of (2). Hence the condition (2) is the strongest among all conditions of the form (3) which hold for all non-trivial DCS (i. e., DCS different from $\{Z\}$). The proof uses product-invariant relations, i. e. the relations which are invariant with respect to all automorphism of the semigroup $(N, .)$.

For every prime p the DCS

$$(4) \qquad 0 \pmod p,\ 1 \pmod p,\ \ldots,\ p-1 \pmod p$$

has the following property:

The union of any subset X of (4), $1 < \mathrm{card}(X) < k$ is not a congruence class (by any modulus).

All DCS (except $\{Z\}$) with this property will be called irreducible DCS, abbreviation IDCS. There are IDCS which are not of the form (4). For example, the congruence classes $0, 4 \pmod 6$, $1, 3, 5, 9 \pmod{10}$, $2 \pmod{15}$, $7, 8, 14, 20, 26, 27 \pmod{30}$ form an IDCS with the common modulus 30 (it is Porubský's example of a nonnatural DCS in essential). In [2] many IDCS are constructed and it is proved that an IDCS with the common modulus n exists if and only if n is a prime (then only (4) can be obtained) or n is divisible by at least three different primes. Further, an operation of splitting is defined which allows to obtain all DCS from the degenerated DCS $\{Z\} = \{0 \pmod 1\}$ and the IDCS. If only IDCS of the form (4) are used then so called natural DCS are exactly obtained.

For every prime p denote $\mathcal{F}(p) = p - 1$, and extend the function $\mathcal{F}$ to the set N by the formula $\mathcal{F}(x.y) = \mathcal{F}(x) + \mathcal{F}(y)$. The Mycielski's conjecture stated $k \geqq 1 + \mathcal{F}(n_i)$ for every DCS (1) and every $i \in \{1, 2, \ldots, k\}$. The main result of 3 is that for all DCS which are not natural (hence e. g. for all IDCS which are not of the form (4)) it holds

$$(5) \qquad k \geqq 6 + \mathcal{F}(n_i)\ .$$

The proof is rather complicated but elementary. The constant 6 in (5) is the best possible. We stated the hypothesis that the modulus n_i in (5) can be replaced by the common modulus of (1).

The IDCS with the common modul pqr (where p, q, r are distinct primes) are completely described, and the number of them is determined, in [4].

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The aim of this, and the subsequent note, is to announce a selection of results presented at the Colloquium on Topology held in Eger in August 1983, and at the Semester of Topology in Banach Center in April 1984. I feel that it is time to prove deeper results about Suslin sets derived from Borel sets in compact spaces.

1. By a space we mean a completely regular T_2 topological space. We denote by $\mathcal{S}(\mathcal{M})$ the collection of Suslin sets derived from the collection of sets $\mathcal{M}$. Recall that $\mathcal{S}(\mathcal{S}(\mathcal{M})) = \mathcal{S}(\mathcal{M}) \supset \mathcal{M}_\sigma \cup \mathcal{M}_\delta$. We denote by $\mathcal{S}_d(\mathcal{M})$ the sets in $\mathcal{S}(\mathcal{M})$ with disjoint Suslin representation. Denote by Σ the space ω^ω with product topology where ω has the discrete topology.

Lemma 1. Let Y be a subset of a space X. Then

(a) $Y \in \mathcal{S}$ (closed(X)) iff some closed set in $X \times \Sigma$ projects onto Y.

(b) $Y \in \mathcal{S}$ (open(X)) iff some open set in $X \times \Sigma$ projects onto Y.

(c) $Y \in \mathcal{S}$ (open(X) $\cup$ closed(X)) (= $\mathcal{S}$ (Borel(X)) iff the intersection of a closed set and a G_δ set in $X \times \Sigma$ projects onto Y.

Note that (a) is classical, and (c) is essentially due to Fremlin [Fre].

2. Theorem 1. The following conditions on a space X are equivalent:

(1a) Some Čech complete subspace of $X \times \Sigma$ projects onto X.

(1b) If X is a subspace of Z then $X \in \mathcal{S}$(Borel(Z)).

(1c) X is obtained by Suslin operation from locally compact sets in some $Z \supset X$.

(1d) There exists a complete sequence of σ-relatively open covers of X.

A space X satisfying the equivalent conditions in Theorem 1 will be called Čech-analytic (following [Fre]). To be sure note that a cover $\mathcal{U}$ of X is called σ-relatively open if $\mathcal{U} = \cup\{\mathcal{U}_n | n \in \omega\}$ such that each $\mathcal{U}_n$ is an open cover of $\cup \mathcal{U}_n$. It was proved in [Ž] that if $X \in \mathcal{S}$(Borel(K)) for some compactification of X, then it holds for any compactification of X. Fremlin [Fre] introduced implicitly (1a) and showed the equivalence with Žolkov's definition. If the space X is hereditarily Lindelöf then (1d) implies that X has a complete sequence of countable covers, and hence it is ω-analytic (= K-analytic in Choquet and Sneider terminology) by [F]. The following result is a solution of a problem of Fremlin.

Theorem 2. A space X is ω-analytic iff it is Čech analytic and there exists an usco-compact correspondence from a separable metric space onto X.

The proof is based on the following

Lemma 2. Let f be a perfect mapping ot X onto a metrizable space Y, and let $\{\mathcal{U}_n\}$ be a sequence of families of open sets in X. There exists a factorization $f = h \circ g$ such that $g: X \longrightarrow S$, $h: S \longrightarrow Y$ are perfect, S is metrizable, and for each n

$\{y \mid g^{-1}y \subset \bigcup \mathcal{U}_n\} = \bigcup\{\{y \mid g^{-1}y \subset U\} \mid U \in \mathcal{U}_n\}$.

3. <u>Theorem 3</u>. The following conditions on a space X are equivalent:

(2a) Some Čech complete subspace of $X \times \Sigma$ injectively projects onto X.

(2b) If X is a subspace of Z then $X \in \mathcal{S}_d(\mathrm{Borel}(Z))$.

(2c) X is obtained by the disjoint Suslin operation from locally compact subsets in some $Z \supset X$.

(2d) There exists a complete sequence $\{\bigcup\{\mathcal{M}_s \mid s \in \omega^n\} \mid n \in \omega\}$ of covers such that each $\mathcal{M}_s$ is an open cover of $M_s = \bigcup \mathcal{M}_s$, $M_s = \bigcup\{M_{si} \mid i \in \omega\}$ for each s, and if $\sigma \in \Sigma$, $M_n \in \mathcal{M}_{\sigma|n}$ then $\bigcap\{\overline{\bigcap\{M_i \mid i \leq n\}} \mid n \in \omega\} \in \bigcap\{M_{\sigma|n} \mid n \in \omega\}$.

A space satisfying the equivalent condition in Theorem 3 will be called Čech-Luzin. Any Čech-Luzin space X is absolutely bi-Suslin (Borel), and I do not know whether or not the converse holds.

The basic stability results follow easily from (ia) and the fact that any countable ($\neq 0$) power of Σ is homeomorphic to Σ.

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DISTINGUISHED SUBCLASSES OF ČECH-ANALYTIC SPACES

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This is a free continuation of [F_3]. Recall that if $\mathcal{F}$ is a set of families of subsets of X then a family $\{X_a \mid a \in A\}$ in X is called $\mathcal{F}$ σ-decomposable if there exist families $\{X_{an} \mid a \in A\}$ in $\mathcal{F}$, $n \in \omega$, such that $X_a = \bigcup\{X_{an} \mid n \in \omega\}$ for each a. So it is clear what is meant by discretely σ-decomposable. We shall call a family $\{X_a\}$ in a topological space uniformly discrete if it is discrete in the finest uniformity inducing the topology. A family $\{X_a\}$ is called isolated if it is discrete in $\bigcup\{X_a\}$.

Following [F-H_1], if $\varkappa$ is an infinite cardinal then a space X is called $\varkappa$-analytic (or topologically $\varkappa$-analytic, abb. T $\varkappa$-analytic) if there exists an usco-compact correspondence from the metric space $\varkappa^\omega$ onto X such that the image of each discrete family (equivalently, discretely decomposable family) is uniformly discretely (or discretely, resp.) σ-decomposable. If the values are disjoint, then the space is called $\varkappa$-Luzin (or topologically $\varkappa$-Luzin, resp.), and if the values are singletons or empty then we speak about point-$\varkappa$-analytic etc. spaces. Analytic means $\varkappa$-analytic for some $\varkappa$, and similarly Luzin etc. The theory of analytic and Luzin spaces was developed in [F-$H_{1,2,3}$]. A discussion of topologically analytic spaces appeared in [H-J-R].

Theory of analytic spaces has two important advantages in comparison with that of topological analytic spaces:

(a) there is a nice description of analytic spaces as Suslin (closed) subsets of products $K \times M$ with K compact and M complete metric.

(b) Using the product $X \times \Sigma$ taken in uniform spaces then the projection $X \times \Sigma \rightarrow X$ preserves uniformly discretely σ-decomposable families.

Lemma 1. If Y is a separable metric space then for any X the projection along Y preserves isolatedly σ-decomposable families.

Lemma 1 is the main point for introducing weakly topologically analytic (abb. WT analytic) spaces as images of complete metric spaces under usco-compact correspondences preserving isolatedly σ-decomposable families. Indeed we have the following characterization.

Theorem 1. Each of the following conditions is necessary and sufficient for X to be WT analytic:

(3a) Some paracompact Čech complete subspace of $X \times \Sigma$ projects onto X.

(3d) There exists a complete sequence of σ-isolated covers.

Of course, analytic or T analytic spaces are characterized by existence of a complete sequence of σ-uniformly discrete or σ-discrete covers, resp.

Theorem 2. Each of the following conditions is necessary and sufficient for X to be WT point-analytic:

(4a) Some completely metrizable subspace of $X \times \Sigma$ projects onto X.

(4d) There exists a complete sequence of σ-isolated covers with clusters of Cauchy filters being singletons.

(4e) X is Čech-analytic and there exists a σ-isolated network for X.

Using the main result of [F-H_1], we obtain

Theorem 3. In a WT point-analytic space X each point-finite completely $\mathcal{S}$(Borel(X))-additive family is isolatedly σ-decomposable. In WT analytic spaces X the result is true for Suslin (closed(X)) sets.

For the first separation principle the following kind of sets works. For each X let Isol Bo(X) be the smallest collection which contains open and closed sets of X, and which is closed under formation of countable intersections and σ-isolated unions.

There are many reasons for trying to understand whether or not the classes of all WT analytic or Čech analytic spaces are preserved by perfect maps. All I know is:

Theorem 4. The perfect image of a Čech analytic space is analytic if metrizable.

The proof depends on Lemma 2 from [F_3].

Note that analytic spaces are paracompact, T analytic spaces are subparacompact, and WT analytic spaces are σ-isolatedly refinable (also called weakly Θ-refinable spaces).

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