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A NOTE ON REFLECTIVE SUBCATEGORIES DEFINED BY PARTIAL ALGEBRAS

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Abstract: By using a generalized partial F-algebra a full subcategory of a certain comma category will be defined. Then a sufficient condition will be given to provide the reflectivity of this subcategory.

Key words: F-algebra, generalized partial F-algebra, comma category, free completion of a g.p.a.

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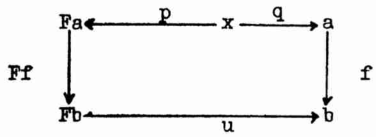
1. Preliminaries. Given an endofunctor $F:\underline{A} \longrightarrow \underline{A}$ on the category $\underline{A}$ one can define the category $\underline{A}(F)$ of F-algebras (see e.g. in [1 - 5]). Let $U:\underline{A}(F) \longrightarrow \underline{A}$ denote the canonical forgetful functor.

A generalized partial F-algebra (a g.p.a.) in the sense of Koubek and Reiterman is a diagram $Fa \xleftarrow{p} x \xrightarrow{q} a$ in $\underline{A}$ (cf. [5]). In the present paper we shall consider the full subcategory $(a{\downarrow}U)^{\langle p,q\rangle}$ of the comma category $(a{\downarrow}U)$ which can be defined in a natural way by means of a g.p.a. Thus the free completion problem for the g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ (see [3, 5]) will be equivalent to the existence of an initial object in $(a{\downarrow}U)^{\langle p,q\rangle}$. The main aim of this note is to establish conditions providing the reflectivity of $(a{\downarrow}U)^{\langle p,q\rangle}$ in $(a{\downarrow}U)$. Since the reflection functor sends $(a{\downarrow}U)$-initial objects to $(a{\downarrow}U)^{\langle p,q\rangle}$-initial ones we shall also obtain criteria for the existence of the free completion.

2. Reflective subcategories in $(a\downarrow U)$. Given a g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ define the objects of the full subcategory $(a\downarrow U)^{\langle p,q\rangle}$ in $(a\downarrow U)$ by requiring the commutativity of (2./1).

(2./1)

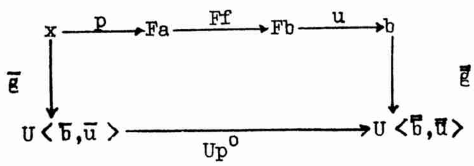


In other words: $\langle\langle b,u\rangle, f\rangle \in |(a\downarrow U)^{\langle p,q\rangle}|$ iff (2./1) commutes. Now we are ready to formulate our main result.

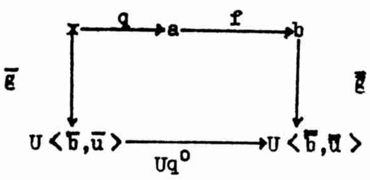
2.1. Theorem. Let $\langle\langle b,u\rangle, f\rangle \in |(a\downarrow U)|$ be an object and suppose that $\mathbf{A}(F)$ has coequalizers of all pairs. If there are initial objects in the comma categories $(x\downarrow U)$ and $(b\downarrow U)$ then there exists an initial object in $(\langle\langle b,u\rangle, f\rangle\downarrow E)$, where E is the natural $(a\downarrow U)^{\langle p,q\rangle} \longrightarrow (a\downarrow U)$ embedding.

Proof. Let $x \xrightarrow{\bar{g}} \bar{b} \xleftarrow{\bar{u}} F\bar{b}$ and $b \xrightarrow{\bar{\bar{g}}} \bar{\bar{b}} \xleftarrow{\bar{\bar{u}}} F\bar{\bar{b}}$ represent initial objects in $(x\downarrow U)$ and $(b\downarrow U)$ respectively. Clearly, there exist unique $\mathbf{A}(F)$-morphisms $p^o, q^o: \langle\bar{b},\bar{u}\rangle \longrightarrow \langle\bar{\bar{b}},\bar{\bar{u}}\rangle$ and $r:$ $: \langle\bar{\bar{b}},\bar{\bar{u}}\rangle \longrightarrow \langle b,u\rangle$ making the diagrams (2./2-4) commute.

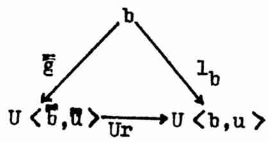
(2./2)



(2./3)



(2./4)



Form the coequalizer $\langle \bar{b},\bar{u}\rangle \overset{rop^o}{\underset{roq^o}{\rightrightarrows}} \langle b,u\rangle \xrightarrow{e^o} \langle b^o,u^o\rangle$ in $\underline{A}(F)$.

We claim that $e^o: \langle\langle b,u\rangle,f\rangle \longrightarrow E\,\langle\langle b^o,u^o\rangle, e^o of\rangle$ is initial in $(\langle\langle b,u\rangle,f\rangle \downarrow E)$. $u^o o(Fe^o)o(Ff)op = e^o ouo(Ff)op =$
$= e^o oro\bar{\bar{g}}ouo(Ff)op = e^o orop^o o\bar{g} = e^o oroq^o o\bar{g} = e^o oro\bar{\bar{g}}ofoq =$
$= e^o ofoq$ proves that $\langle\langle b^o,u^o\rangle, e^o of\rangle \in |(a\downarrow U)^{\langle p,q\rangle}|$ as required.
For a morphism $e':\langle\langle b,u\rangle,f\rangle \longrightarrow E\langle\langle b',u'\rangle,f'\rangle$ in $(a\downarrow U)$ we have
$(U(e'orop^o))o\bar{g} = e'orop^o o\bar{g} = e'oro\bar{\bar{g}}ouo(Ff)op = e'ouo(Ff)op =$
$= u'o(Fe')o(Ff)op = u'o(Ff')op = f'oq = e'ofoq = e'oro\bar{\bar{g}}ofoq =$
$= e'oroq^o o\bar{g} = (U(e'oroq^o))o\bar{g}$.
The $(x\downarrow U)$ initiality of $\langle\langle \bar{b},\bar{u}\rangle,\bar{g}\rangle$ immediately gives that
$e'orop^o = e'oroq^o$. Hence there is a unique $\underline{A}(F)$-morphism t:
: $\langle b^o,u^o\rangle \longrightarrow \langle b',u'\rangle$ with $toe^o = e'$. But easily can be seen
that for a morphism $t: \langle b^o,u^o\rangle \longrightarrow \langle b',u'\rangle$ the condition $toe^o =$
$= e'$ is equivalent to the commutativity of (2./5). ‖

$$(2./5)\qquad \begin{array}{ccc} & & E\langle\langle b^o,u^o\rangle, e^o o f\rangle \\ & \overset{e^o}{\nearrow} & \big\downarrow Et \\ \langle\langle b,u\rangle, f\rangle & & \\ & \underset{e'}{\searrow} & \\ & & E\langle\langle b',u'\rangle, f'\rangle \end{array}$$

The next theorem is an obvious consequence of 2.1.

2.2. Theorem. Let $\underline{A}(F)$ have free algebras (i.e. $\dashv U$) and coequalizers of all pairs. Then

(i) for each g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ the full subcategory $(a{\downarrow}U)^{\langle p,q\rangle}$ is reflective in $(a{\downarrow}U)$;

(ii) each g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ has a free completion in $\underline{A}(F)$.

2.3. Remark. The (ii) part of the above theorem improves a result of Koubek and Reiterman ([5] p. 220). Indeed, if $\underline{A}$ is cocomplete, E-co-well-powered and $F:\underline{A} \longrightarrow \underline{A}$ preserves E of an image factorization system (E,M), then $\underline{A}(F)$ has coequalizers of all pairs (see [1 - 3]).

2.4. Remark. The reflection of an $(A{\downarrow}U)$-object $a \xrightarrow{f} b \xleftarrow{u} Fb$ in $(a{\downarrow}U)^{\langle p,q\rangle}$ also can be obtained by using a certain free completion. Take the g.p.a. $Fb \xleftarrow{\tilde{p}} x \amalg Fb \xrightarrow{\tilde{q}} b$ where $x \amalg Fb$ denotes an $\underline{A}$-coproduct with injections j_x, j_{Fb} and $\tilde{p} o j_x = (Ff)op$, $\tilde{p} o j_{Fb} = 1_{Fb}$ defines $\tilde{p}$ and $\tilde{q} o j_x = foq$, $\tilde{q} o j_{Fb} = u$ defines $\tilde{q}$. The free completion (2./6) of this g.p.a. yields the required reflection: $k: \langle\langle b,u\rangle, f\rangle \longrightarrow \langle\langle b^o,u^o\rangle, kof\rangle$.

$$(2./6) \qquad \begin{array}{ccccc} Fb & \xleftarrow{\tilde{p}} & x \sqcup Fb & \xrightarrow{\tilde{q}} & b \\ {\scriptstyle Fk}\downarrow & & & & \downarrow{\scriptstyle k} \\ Fb^{o} & & \xrightarrow[u^{o}]{} & & b^{o} \end{array}$$

R e f e r e n c e s

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