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ON CERTAIN EINSTEIN SPACE-TIME

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Abstract: The subject of the present note is the Riemannian space-time provided with the pseudo-metric tensor (1).

Key words: Riemannian space, pseudo-metric tensor.

Classification: 53C50, 53C80

In the work [1] a classification of four-dimensional generalized symmetric pseudo-Riemannian spaces has been carried out.

It appears that there are four such spaces and among them only one is of signature (+ + + -), namely that which is provided with the following pseudo-metric tensor on the Cartesian space R^4:

$$(1) \qquad (g_{\mu\lambda}) = \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

A four-dimensional pseudo-Riemannian space of signature (+ + + -) is called a Riemannian space-time.

Theorem 1. The Riemannian space-time with the metric (1) is an Einstein space-time of zero Ricci tensor.

Proof: Computing Christoffel symbols from formula (1) we have

$$(2)\qquad \begin{aligned} &\Gamma^3_{11} = -2\,e^{2t} \qquad &&\Gamma^3_{22} = 2\,e^{-2t} \\ &\Gamma^1_{14} = 1 \qquad &&\Gamma^2_{24} = -1, \text{ remaining } \Gamma^{\nu}_{\lambda\mu} = 0. \end{aligned}$$

Hence we obtain the following curvature tensor:

$$(3)\qquad \begin{cases} R^3_{141} = 2\,e^{2t}, \qquad & R^1_{144} = 1 \\ R^3_{242} = 2\,e^{-2t}, \qquad & R^2_{244} = -1 \\ \text{remaining } R^{\nu}_{\alpha\beta\mu} = 0. \end{cases}$$

On the basis of formula (3) it is easy to check that Ricci tensor is zero:

$$R_{\lambda\mu} = 0.$$

It is sometimes important to give the Petrov-Penrose type [2] for an Einstein space-time. For this purpose, a subclassification to the ranks r_R, $\overset{2}{r}_R$, $\tilde{r}_R$ of the curvature tensor ([3]; Theorem 4, page 52) has been carried out:

(4)

$\overset{(+2)}{\underset{(H)}{G}}_{abc}$ \ $\overset{*}{T}_1, T_1$	$\overset{*}{T}_1$	T_1	$\overset{*}{T}_2$	T_2	$\overset{*}{T}_3$	T_3
$\overset{(+2)}{\underset{(H)}{G}}_{466}$	I | D | O	I | D	II | N	II	III	
$\overset{(+2)}{\underset{(H)}{G}}_{464}$	I | D	I	II			III
$\overset{(+2)}{\underset{(H)}{G}}_{442}$	D					
$\overset{(+2)}{\underset{(H)}{G}}_{342}$				N		
$\overset{(+2)}{\underset{(H)}{G}}_{000}$		M				

Here the spaces $\overset{*}{T}_i$, T_i are of Petrov type, spaces I, D, O, II, III, N, M are Penrose subtypes, and empty places mean that the corresponding type does not exist, and that $r_R = a$, $\overset{2}{r}_R = b$, $\tilde{r}_R = c$.

On the basis of (3) we have:

$$(5) \qquad r_R = 3, \qquad \tilde{r}_R = 2.$$

According to (5) and (4), our space is of the $\underset{(H)}{G_{342}}$ type, which determines the Petrov-Penrose type N.

Hence we have the following:

<u>Theorem 2</u>. The Einstein space-time with the metric (1) is of Petrov-Penrose type N.

For this type, the tensor (1) is interpreted as a pure field of gravity. For the field of gravity (1) we find the equations of motion, i.e. geodesic curves:

$$(6) \qquad \begin{aligned} &a) \quad \ddot{\xi}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\, \dot{\xi}^{\alpha} \dot{\xi}^{\beta} = 0 \\ &b) \quad g_{\alpha\beta}\, \dot{\xi}^{\alpha} \dot{\xi}^{\beta} = A \ (A = 0 \text{ or } A \neq 0, \text{ A-constant}). \end{aligned}$$

Substituting (2) and (1) in (6) we get the following system of ordinary differential equations:

$$(7) \qquad \begin{cases} \ddot{x} + 2\,\dot{x}\,\dot{t} = 0 \\ \ddot{y} - 2\,\dot{y}\,\dot{t} = 0 \\ \ddot{z} - 2\,e^{2t}\,\dot{x}^2 + 2\,e^{-2t}\,\dot{y}^2 = 0 \\ \ddot{t} = 0 \Longrightarrow \dot{t} = k \ (= \text{const.}) \\ e^{2t}\,\dot{x}^2 + \dot{z}\,\dot{t} + e^{-2t}\,\dot{y}^2 = A. \end{cases}$$

Assuming the time t to be a parameter for the geodesic curves determined by the system of equations (7) we get:

$$(8)\quad \begin{cases} \ddot{x} + 2\,k\,\dot{x} = 0 \\ \ddot{y} - 2\,k\,\dot{y} = 0 \\ \ddot{z} - 2\,e^{2t}\,\dot{x} + 2\,e^{-2t}\,\dot{y}^2 = 0 \\ e^{2t}\,\dot{x}^2 + k\,\dot{z} + e^{-2t}\,\dot{y}^2 = A. \end{cases}$$

Solving the system of equations (8) we have the following formulas for geodesic curves:

$$(9)\quad \begin{cases} x = C_1 + C_2\,e^{-2t} \\ y = C_3 + C_4\,e^{2t} \\ z = 2\,C_2^2\,e^{-2t} - 2\,C_4^2\,e^{2t} + A\,t + B \\ t = t \end{cases}$$

For the motion of a probe particle in a field of gravity (1) with the initial conditions

$$\begin{pmatrix} (x_o,\ y_o,\ z_o,\ 0) \\ (\dot{x}_o,\ \dot{y}_o,\ \dot{z}_o,\ 1) \end{pmatrix}$$

we have the following formulas:

$$(10)\quad \begin{cases} x = x_o + \frac{1}{2}\,\dot{x}_o - \frac{1}{2}\,\dot{x}_o\,e^{-2t} \\ y = y_o - \frac{1}{2}\,\dot{y}_o + \frac{1}{2}\,\dot{y}_o\,e^{2t} \\ z = \frac{1}{2}\,\dot{x}_o^2\,e^{-2t} - \frac{1}{2}\,y_o\,e^{2t} + (\dot{z}_o + \dot{x}_o^2 + \dot{y}_o^2)\,t + z_o - \\ \qquad - \frac{1}{2}\,\dot{x}_o^2 + \frac{1}{2}\,\dot{y}_o^2 \\ t = t \quad (t \geq 0). \end{cases}$$

On the basis of [1, Theorem 5.2] and our Theorem 2 we get finally

<u>Theorem 3</u>. There is one and only one Einstein time-space of type N, which is at the same time a generalized symmetric space, namely that provided with the pseudo-metric tensor (1).

R e f e r e n c e s

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