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ADDENDUM TO THE PAPER „SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS"

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Abstract: Let E be a Banach space, M a compact metric space, K a nonempty closed convex subset of E, and T a continuous mapping from K into M. If F is a K_Φ-mapping from $M \times K$ to 2^K ([5]), then there is a point x_0 in K such that $x_0 \in F(Tx_0, x_0)$ Here we give an application of this result to the theory of differential relations.

Key words: Multivalued mappings, fixed points, Banach spaces, differential relations.

Classification: 54C60, 47H10

Let $\mathcal{X}(X)$ denote the family of all nonempty closed convex bounded subsets of a normed linear space X. The set $\mathcal{X}(X)$ will be regarded as a metric space endowed with the Hausdorff distance d_X, i.e.

$$d_X(A,B) = \max [\sup_{x \in A} d(x,B), \sup_{x \in B} d(x,A)]$$

for $A, B \in \mathcal{X}(X)$; here the distance between any point $x \in X$ and subset Q of X is denoted by $d(x,Q)$.

Let $(E, \|\cdot\|)$ be a uniformly convex Banach space, M a compact metric space, K a nonempty closed convex subset of E, T a single-valued mapping from K into M, and F a mapping from $M \times K$ to $\mathcal{X}(X)$. Let us suppose that:

(1) T is continuous on K,

(2) $F(\cdot, x)$ is continuous on M for every $x \in K$, and

(3) $d_K(F(x,y_1), F(x,y_2)) \leq k \| y_1 - y_2 \|$ for all $x \in M$ and $y_1, y_2 \in K$ and with a constant $k < 1$. Under these hypotheses there exists a point x_0 in K such that $x_0 \in F(Tx_0, x_0)$.

The proof of this theorem resembles that of [5] and therefore will be omitted. Our result has applications, whose basic ideas are illustrated by the example below.

Example. Let $I = [0,a]$ and $J = [0,h]$ $(0 < h \leq a)$. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $L^2(J, \mathbb{R}^n)$ the Banach space of measurable functions from J to $\mathbb{R}^n$ such that $\|x\| = (\int_0^h |x(t)|^2 dt)^{1/2} < \infty$, and $C(J, \mathbb{R}^n)$ the Banach space of continuous functions from J to $\mathbb{R}^n$ with the usual supremum norm.

We follow here the terminology of [1] and [3]. Suppose that $f: I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathfrak{X}(\mathbb{R}^n)$ is a mapping satisfying the following conditions:

(i) $t \longmapsto f(t,u,v)$ is measurable on I for each fixed u,v in $\mathbb{R}^n$, and $(u,v) \longmapsto f(t,u,v)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ for each fixed $t \in I$;

(ii) there exists $m \in L^2(I, \mathbb{R})$ such that $d_{\mathbb{R}^n}(f(t,u,v), \{\theta\}) \leq m(t)$ for $t \in I$ and u, v in $\mathbb{R}^n$ (θ denote the zero of the space $\mathbb{R}^n$);

(iii) $d_{\mathbb{R}^n}(f(t,u,v_1), f(t,u,v_2)) \leq L|v_1 - v_2|$ for $t \in I$ and u, v_1, v_2 in $\mathbb{R}^n$, where $L \geq 0$ is a constant.

We define:

$(Tx)(t) = \int_0^t x(s(ds$ for $x \in L^2(J, \mathbb{R}^n)$,

$K = \{x \in L^2(J, \mathbb{R}^n) : |x(t)| \leq m(t)$ a.e. in $J\}$.

Evidently, K is a closed convex bounded subset of $L^2(J, \mathbb{R}^n)$, T is continuous as a map of K into $C(J, \mathbb{R}^n)$, and $T[K]$ is conditionally compact.

If $x \in C(J, \mathbb{R}^n)$ and $y \in K$, then the mapping $t \longmapsto f(t,x(t), (Ty)(t))$ is measurable and therefore has a measurable selector by Kuratowski and Ryll-Nardzewski [4]. Define $F: C(J, \mathbb{R}^n) \times K \longrightarrow \mathfrak{X}(K)$ as follows: $F(x,y)$ is the set of all measurable selectors of $f(\cdot, x(\cdot), (Ty)(\cdot))$.

Let $x \in C(J, \mathbb{R}^n)$ and $y_1, y_2 \in K$, and assume that $w_1 \in F(x,y_1)$. By Hermes [2] (see [1], Lemma 2.5), there exists a measurable selector w_2 of $f(\cdot, x(\cdot), (Ty_2)(\cdot))$ such that

$$|w_1(t) - w_2(t)| = d(w_1(t), f(t,x(t), (Ty_2)(t))$$

on J. Thus, $w_2 \in F(x,y_2)$ and

$$|w_1(t) - w_2(t)| \leq$$

$$\leq d_{\mathbb{R}^n}(f(t,x(t),(Ty_1)(t)), f(t,x(t),(Ty_2((t)) \leq$$

$$\leq L|(Ty_1)(t) - (Ty_2)(t)| \leq$$

$$\leq L \int_0^h | y_1(s) - y_2(s)| ds \leq$$

$$\leq L \sqrt{h} \| y_1 - y_2 \|$$

for $t \in J$. This implies that $\| w_1 - w_2 \| \leq Lh \| y_1 - y_2 \|$. Arguing again as above, it follows that if $w_2 \in F(x,y_2)$ then there exists $w_1 \in F(x,y_1)$ with $\| w_1 - w_2 \| \leq Lh \| y_1 - y_2 \|$.

Consequently, $d_K(F(x,y_1), F(x,y_2)) \leq Lh \| y_1 - y_2 \|$ for $x \in C(J, \mathbb{R}^n)$ and $y_1, y_2 \in K$. Moreover, modifying our reasoning, we obtain that $x \longmapsto F(x,y) (y \in K)$ is a continuous mapping from $C(J, \mathbb{R}^n)$ to $\mathfrak{X}(K)$.

Assume in addition that $Lh < 1$. Now, applying our result to the space $L^2(J, \mathbb{R}^n)$ and the mapping T, F, we infer that there is y_o in K such that

$$y_o(t) \in f(t, \int_0^t y_o(s)ds, \int_0^t y_o(s)ds)$$

for t in J.

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Added in proof. When this paper was already submitted, the author happened to read the work by M. KISIELEWICZ, Generalized functional-differential equations of neutral type, Ann. Polon. Math, XLII(1983), 139-148.

Let A be a nonempty closed convex bounded subset of the Hilbert space Y, Γ an operator with domain A and range in the Banach space X, and G a mapping from $A \times \Gamma[A]$ to the standard space of all nonempty closed convex subsets of A. In his Theorem 2.4, Kisielewicz proved that if $G(\cdot,y)$ is a contraction uniformly with respect to $y \in \Gamma[A]$, $G(x,\cdot)$ is continuous on $\Gamma[A]$ in the relative topology and Γ is completely continuous, then there exists x in A such that $x \in G(x, \Gamma x)$.