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MORSE-SARD THEOREM FOR CLOSED GEODESICS
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Abstract: The existence of finite or infinite number of closed geodesics on a compact Riemannian manifold can be proved under suitable assumptions. The paper brings another type of information. It is proved here that the set Γ of lengths of all closed geodesics on a real-analytic compact Riemann manifold is always a discrete set. The proof is based on a version of Morse-Sard theorem for real-analytic maps.

Key words: Closed geodesics, Morse-Sard theorem.

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The classical Morse theorem says that the set Γ of critical levels of a function $f \in C^k : R^n \longrightarrow R$

$$\Gamma = \{f(x) \mid \nabla f(x) = 0\}$$

has Lebesgue measure zero; $\mathcal{H}_1(\Gamma) = 0$, if $k \geq n$ [1]. The more refined version [2] asserts that $\mathcal{H}_{n/k}(\Gamma) = 0$, where $\mathcal{H}_{n/k}$ is a Hausdorff measure of dimension $n/k \leq 1$. If $f \in C$, we obtain $\mathcal{H}_\varepsilon(\Gamma) = 0$, $\forall \varepsilon > 0$. If the function f is analytic, we can obtain a better estimate of Γ . Namely, [3], Γ is locally finite; i.e. for every compact set $K \subset R^n$, the set

$$\Gamma_K = \{f(x) \mid \nabla f(x) = 0, \ x \in K\}$$

is finite. Clearly, then Γ is denumerable.

Analogous theorems hold for a functional defined on a Banach space if its second derivative is a Fredholm map (and so-

me other. , rather technical hypotheses are satisfied, see
[4 - 7]).

In this paper we prove an analogous theorem for closed ge-
odesics, the functional being the length of a curve.

Theorem 1. Let M be an n-dimensional compact real-analy-
tic Riemannian manifold. Then the set Γ of lengths of all clo-
sed geodesics on M is a discrete set.

Remark. The question of existence of closed geodesics was
extensively studied (see e.g.[9]), and the existence of infini-
te number of them can be proved for some manifolds. The theorem
1 brings on the other hand an upper bound on the number of
lengths of closed geodesics.

The real-analyticity means that there is an atlas of charts
covering M, such that transition maps are real-analytic and the
coefficients of a metric written in these charts are real-ana-
lytic, too. In the same way one may prove that when M is only
a C^{∞} manifold, the set Γ has $\mathcal{H}_{\varepsilon}(\Gamma) = 0$, $\forall \varepsilon > 0$.

The set of closed curves in M does not form a linear spa-
ce, and therefore the theorems of [7] cannot be applied direct-
ly. But this set can be endowed with a structure of a Hilberti-
an manifold [8, 9].

Proof. We shall divide it into three steps. In the first
step we recall the definition and some properties of the Hil-
bertian manifold of curves. In the second step we prove the the-
orem using Lemma 1. In the third step we give a rather techni-
cal proof of this lemma.

Step 1. Let $S = [0,1] / \{0,1\}$ be the unit circle and let
us define the set of closed H^1-curves

$$\Lambda M = \{c : S \to M \mid \int_0^1 \langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)} \, dt < \infty \}$$

where $\langle \cdot , \cdot \rangle_x$ denotes a Riemannian metric on $T_x M$. Let $c_0 : S \to M$ be a fixed C^∞-curve. Let us consider the set of all H^1-vector fields on c_0 (parametrized by $t \in S$)

$$H^1(c_0^* TM) = \{\xi : S \to TM \mid \xi(t) \in T_{c_0(t)} M, \ \langle \xi , \xi \rangle_1 < \infty \}$$

where

$$\langle \xi , \eta \rangle_0 = \int_0^1 \langle \xi(t), \eta(t) \rangle_{c_0(t)} \, dt,$$

$$\langle \xi , \eta \rangle_1 = \langle \xi , \eta \rangle_0 + \langle \nabla \xi , \nabla \eta \rangle_0 \, ;$$

the covariant derivative being taken along the curve

$$(\nabla \xi)(t) = (\nabla_{\dot{c}_0(t)} \xi)(t).$$

There exists an $\varepsilon > 0$ such that [8, 9] the exponential map

$$(\tau , \exp) : TM \to M \times M \ (x, v_x) \mapsto (x, \exp_x v_x), \ v_x \in T_x M$$

is a diffeomorphism, when restricted to the set

$$T^\varepsilon M = \{(x, v_x) \in TM \mid \| v_x \| < \varepsilon \} .$$

Here $\exp_x : T_x M \to M$ is the standard exponential map. Let us define

$$\exp_{c_0} : H^{1, \varepsilon}(c_0^* TM) \to \Lambda M$$

$$\xi = (\xi(t))_{t \in S} \mapsto c = (c(t))_{t \in S}$$

$$c(t) = \exp_{c_0(t)} \xi(t),$$

where

$$H^{1, \varepsilon}(c_0^* TM) = \{\xi \in H^1(c_0^* TM) \mid \| \xi(t) \| < \varepsilon , \ \forall t \in S \}.$$

Then the atlas of charts [8, 9]

$$\{ \exp_{c_0}^{-1} \mid c_0 \in C^\infty(S, M) \}$$

gives to ΛM the structure of a Hilbertian manifold modelled on Hilbert spaces $H^1(c_0^* TM)$. Moreover, $H^1(c_0^* TM)$ is the tangent space $T_{c_0} \Lambda M$.

The energy of a closed curve $c \in \Lambda M$ is defined by

$$E(c) = \frac{1}{2} \int_0^1 \langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)} dt.$$

Then [8, 9] E is C^∞, its Gateâux differential is given by

$$dE(c; \eta) = \int_0^1 \langle \dot{c}, \nabla \eta \rangle dt$$

and $dE(c; \cdot) = 0$ iff c is a constant map or c is a closed geodesic. Grad $E(c) \in T_c \Lambda M$ is defined by

$$\langle \text{grad } E(c), \eta \rangle_1 = dE(c; \eta), \quad \forall \eta \in T_c \Lambda M.$$

The following basic properties of the energy functional are known [8, 9]:

(i) (so-called Palais-Smale condition.)

If $c_m \in \Lambda M$, $E(c_m) \leq K$, $K > 0$ and $\| \text{grad } E(c_m) \|_1 \to 0$, then there is a subsequence of $\{c_m\}$ converging in ΛM to a closed geodesic c_0.

(ii) Let c_0 be a closed geodesic. Then the Hessian A_{c_0} of E at c_0, defined by

$$A_{c_0} : T_{c_0} \Lambda M \to T_{c_0} \Lambda M,$$

$$\langle A_{c_0} \xi, \eta \rangle_1 = d^2 E(c_0; \xi, \eta), \quad \forall \xi, \eta \in T_{c_0} \Lambda M,$$

has the form $A_{c_0} = \text{identity} + \text{compact map}$.

Step 2. Let us suppose that there are closed geodesics c_m such that $E(c_m) \to K < \infty$, $E(c_m) \neq E(c_k)$, $\forall k \neq m$. Using (i) we can suppose that $c_m \to c_0$ in ΛM, c_0 is a geodesic, and that $\{c_m\}$ is in the range of the chart $\exp_{c_0}$. Let us transport the energy functional on the tangent space

$$\bar{E}(\xi) = E(\exp_{c_0}(\xi)), \quad \forall \xi \in T_{c_0}^\varepsilon \Lambda M = H^{1, \varepsilon}(c_0^* TM).$$

Then $\xi_m = \exp_{c_o}^{-1}(c_m)$ are critical points of $\overline{E}$ and $\|\xi_m\|_1 \to 0$.
We want now to apply the infinite dimensional version of the
Morse-Sard theorem [7, Theor. 5.1] to $\overline{E}$. But to prove the analy-
ticity of $\overline{E}$, the regularity properties must be used. We follow
the method of [7, p. 256]. Let us define spaces

$$X_1 = H^2(c_o^* TM) = \{\xi \in H^1(c_o^* TM) \mid \langle \xi, \xi \rangle_2 < \infty\},$$

$$\langle \xi, \eta \rangle_2 = \langle \xi, \eta \rangle_1 + \langle \nabla\nabla\xi, \nabla\nabla\eta \rangle_o ,$$

$$X_2 = H^o(c_o^* TM) = \{\xi : S \to TM \mid \xi(t) \in T_{c_o(t)}M, \langle \xi, \xi \rangle_o < \infty\}$$

and consider grad $\overline{E}$ as a map $F : X_1 \to X_2$ defined by

$$d\overline{E}(\xi ; \eta) = \langle \eta, F(\xi) \rangle_o , \quad \forall \eta \in X_1.$$

By the standard regularity argument (similar to that used in the
proof of the regularity of geodesics) we obtain that

$$\xi_m \in X_1, \quad \|\xi_m\|_2 \to 0.$$

Lemma 1. The functional $\overline{E} : X_1 \to R$ and the operator $F : X_1 \to X_2$ are real-analytic.

Once we have this lemma, all hypotheses of the theorem 5.1
from [7] are fulfilled and it follows that for a sufficiently
large m we have

$$E(c_m) = \overline{E}(\xi_m) = \overline{E}(0) = E(c_o).$$

Hence the critical levels of E are isolated. Now, it suffices
to show that the energy and the length of a geodesic are inter-
related. If c is a critical point of E, then using variations
generated by the reparametrizations of c, we can find that
$\|\dot{c}(t)\| = \omega = $ constant. Then $E(c) = \frac{1}{2}\omega^2$, while

$$L(c) = \int_0^1 \langle \dot{c}, \dot{c} \rangle^{\frac{1}{2}} dt = \omega ,$$

so that $L(c) = (2E(c))^{\frac{1}{2}}$.

<u>Step 3</u>. We shall prove Lemma 1 using the method of [7, p. 256]. We must show that for each $\xi_0 \in X_1$, $\|\xi_0\|_2 < \varepsilon$ ($\varepsilon > 0$ sufficiently small), there is a $\sigma > 0$, such that for every $\chi \in X_1$, $\|\chi\|_2 < \sigma$ the Taylor series

$$\overline{E}(\xi_0 + \chi) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n \overline{E}(\xi_0; \chi, \ldots, \chi)$$

converges. It suffices to show that $\overline{E}$ is a restriction (to X_1) of a map $\widetilde{E}: X_1 + iX_1 \longrightarrow X_2 + iX_2$, which is locally bounded and Gateâux differentiable [7, Theor. 3.7]. Let $I \subset S$ be an interval such that $c_0(I) \subset U$, U is a co-ordinate neighborhood in M (S may be written as a finite union of such intervals). Let us define

$$\overline{E}_I(\xi) = \frac{1}{2} \int_I \langle \dot{c}, \dot{c} \rangle \, dt, \quad c(t) = \exp_{c_0(t)}(\xi(t)).$$

If $(x_1, \ldots, x_n)$ is a co-ordinate system on U we denote by $\overline{x}(x) = (x_1(x), \ldots, x_n(x))$ the co-ordinates of the point $x \in U$. Generally, the bar will denote n-tuples, so that

$$x_i(c_0(t)) = c_{0i}(t), \quad \overline{c}_0(t) = (c_{01}(t), \ldots, c_{0n}(t)).$$

For $\xi \in T_x M$, $x \in U$ we shall write

$$\xi = \xi_i \left. \frac{\partial}{\partial x_i} \right|_x, \quad \overline{\xi} = (\xi_1, \ldots, \xi_n),$$

$$x_i(\exp_x \xi) = e_i(\overline{x}(x); \overline{\xi}).$$

Using the summation convention we can write

$$\frac{d}{dt} \exp_{c_0} \xi = h_i(t, \overline{\xi}, \dot{\overline{\xi}}) \left. \frac{\partial}{\partial x_i} \right|_{\exp_{c_0} \xi},$$

$$h_i(t, \overline{\xi}, \dot{\overline{\xi}}) = \frac{d}{dt} e_i(\overline{c}_0; \overline{\xi}) =$$

$$= \frac{\partial e_i}{\partial c_{0k}}(\overline{c}_0; \overline{\xi}) \dot{c}_{0k} + \frac{\partial e_i}{\partial \xi_k}(\overline{c}_0; \overline{\xi}) \dot{\xi}_k,$$

where the dependence of $c_0, \xi, \overline{c}_0, \overline{\xi}, \dot{\overline{\xi}}$ on t was supressed. Using the co-ordinate form g_{ij} of the metric $\langle \cdot, \cdot \rangle$ we have

$$\bar{E}_I(\xi) = \frac{1}{2} \int_I f(t, \bar{\xi}, \dot{\bar{\xi}}) \, dt,$$

where

$$f(t, \bar{\xi}, \dot{\bar{\xi}}) = g_{ij}(\bar{e}(\bar{c}_o; \bar{\xi})) h_i(t, \bar{\xi}, \dot{\bar{\xi}}) h_j(t, \bar{\xi}, \dot{\bar{\xi}}).$$

The functions e_i are real-analytic [10] and g_{ij} are real-analytic, too. Thus, they are restrictions (to real values of their arguments) of holomorphic functions $\tilde{e}_i$, $\tilde{g}_{ij}$ resp. [11; 7, Theor. 3.11]. Then functions h_i and f are also restrictions of $\tilde{h}_i$, $\tilde{f}$ defined for $\bar{\xi}$, $\dot{\bar{\xi}}$ complex vectors from the neighborhood of allowed real values of $\bar{\xi}$, $\dot{\bar{\xi}}$, i.e. $|\bar{\xi}|$, $|\dot{\bar{\xi}}| < \varepsilon$ (the variable t remains always real). At this moment we need the regularity, because we must know that $|\bar{\xi}|_{C^1} < \varepsilon$; this follows from the embedding $X_1 \subset C^1$. Clearly, h_i and f are C^∞ in t and we can apply Lemma 3.1 from [7, App. VI] which gives us the analyticity of $\bar{E}_I$. Equivalently, we can verify that

$$\tilde{E}_I(\bar{\xi}) = \frac{1}{2} \int_I \tilde{f}(t, \bar{\xi}, \dot{\bar{\xi}}) \, dt, \quad \bar{\xi}, \dot{\bar{\xi}} \in C^n$$

is locally bounded and Gateâux differentiable. The proof of the analyticity of F_I is similar to the proof of Lemma 3.1 [7, App. VI]. Making a finite sum of E_I's (resp. F_I's) we obtain the analyticity of $\bar{E}$ and F.

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