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PSEUDOCOMPACT SPACES WITH A σ-POINT-FINITE BASE ARE METRIZABLE

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Abstract: D.B. Shachmatov has recently constructed an example of a non-metrizable pseudocompact space with a point-countable base. We apply a method due to Stephen Watson to prove that such a base cannot be the countable union of point-finite families.

Key words: Pseudocompact spaces, σ-point-finite cover, Miščenko's theorem, Baire spaces.

Classification: 54D30, 54E35

A.S. Miščenko proved [1] in 1962 that compact Hausdorff spaces with a point-countable base are metrizable. Since spaces with a point-countable base are metalindelöf (= every open cover has a point-countable open refinement) and countably compact metalindelöf spaces are compact, the Miščenko theorem is also true for countably compact Hausdorff spaces. A further generalization to the case of pseudocompact spaces is not possible: D.B. Shachmatov has recently shown [2] that a pseudocompact space with a point-countable base can contain a closed discrete subspace of arbitrary cardinality. In connection with the Shachmatov's example, the question arises whether pseudocompact spaces with a σ-point-finite (= the countable union of point-finite families) base are metrizable. The purpose of this note is to answer this question in the affirmative.

We closely follow the Watson's proof [3] of the Scott-Förster-Watson theorem: pseudocompact metacompact (= every open cover has a point-finite open refinement) spaces are compact. Our proposition is a slight generalization of a result proved (though not explicitly stated) in [3]:

Watson's lemma. Every point-finite open cover of a pseudocompact space X contains a finite subfamily whose union is dense in X.

Proposition. The conclusion of the Watson's lemma remains valid for any σ-point-finite open cover of a pseudocompact space X.

We begin with a known lemma concerning Baire spaces. A space is Baire if any countable union of nowhere dense sets has an empty interior.

Lemma. ([4],[5].) Every point-finite family P of open subsets of a Baire space X is locally finite at a dense set of points.

An apparently stronger version of this lemma can be found in [3]. It is noticed in [4, Theorem 4] and [5, Theorem 3.10] that the property of a space X stated in the lemma is in fact equivalent to the Baire property.

Let us sketch the proof. For every natural n, let $X_n = \{x \in X$: x is in at most n elements of $P\}$. Let Y be the union of the boundaries of the sets X_n, n = 0,1,... . Since X is Baire, the interior of Y is empty. The family P is easily seen to be locally finite at each point of the set $X \setminus Y$ which is dense in X.

Proof of the proposition. Let P be a σ-point-finite open cover of a pseudocompact space X. Choose an increasing sequence $P_1 \subseteq P_2 \subseteq \dots$ of point-finite families such that $P = \bigcup \{P_n : n = 1,2,\dots\}$. Let B_n be the collection of nonempty open subsets $V \subseteq X$ such that the set $P_n(V) = \{U \in P_n : U \cap V \neq \emptyset\}$ is finite. By the lemma (which is applicable here, since pseudocompact spaces are Baire), each B_n is a π-base for X. Suppose that for any finite subset $Q \subseteq P$ the union of Q is not dense in X. Then a sequence $V_1 \in B_1$, $V_2 \in B_2, \dots$ can be defined by induction so that each V_n is contained in $X \setminus \overline{\bigcup \{P_k(V_k) : 1 \leq k < n\}}$ (this set is not empty by our assumption, since each $P_k(V_k)$ is a finite subset of P). If $U \in P_m$ meets some V_k with $k \geq m$, then $U \in P_k(V_k)$ and $U \cap V_n = \emptyset$ for any $n > k$. Hence each $U \in P$ meets only finitely many members of the sequence $\{V_n : n = 1,2,\dots\}$. It follows that this sequence is locally finite, in contradiction with the pseudocompactness of X. The proposition is proved.

The theorem stated in the title now readily follows: the proposition implies that any (completely regular) pseudocompact space with a σ-point-finite base is compact and therefore metrizable by the Miščenko's theorem.

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