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# A LARGE $F_\sigma$ -DISCRETE FRÉCHET SPACE HAVING THE SOUSLIN PROPERTY

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Abstract: By a theorem of G. Amirdzhanov, any $\sigma$-product of closed unit intervals (= the subspace of a Tychonoff cube consisting of all points having only finitely many non-zero coordinates) contains a dense subspace of countable pseudocharacter. We give a simple proof of a more general fact: any such $\sigma$-product contains a dense subspace which is the union of countably many closed discrete sets and therefore has a $G_\delta$ diagonal. This answers a question (first answered by D.B. Shachmatov) raised by P. Simon, J. Ginsburg and R.G. Woods of whether a regular space having a $G_\delta$ diagonal and the Souslin property can be of cardinality greater than exp $\aleph_0$.

Key words: $G_\delta$ diagonal, Souslin number, pseudocharacter, Fréchet space, countable tightness, $\sigma$-product, $F_\sigma$-discrete.

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Consider the following four cardinal invariants of a topological space X: (1) the Lindelöf number $l(X)$; (2) the Souslin number $c(X)$; (3) the character $\chi(X)$; and (4) the product $\psi(X)\cdot t(X)$ of the pseudocharacter and the tightness. The first two invariants are "global", the last two are "local". Suppose one of the "global" invariants and one of the "local" invariants of a Hausdorff space X do not exceed a given cardinal m. Is it true that X cannot be too large? It is well known that the answer is yes for three of the four possible combinations: (1) and (3) (Arhangel´skii), (1) and (4) (Arhangel´skii (for a regular X) - R. Pol - Shapirovskii), (2) and (3) (Hajnal -

Juhász). In these three cases the cardinality of X does not exceed exp m, see e.g. [1],[2]. In the fourth case, when c, $\psi$ and t are bounded, the cardinality can be as great as one chooses it to be: for any cardinal m, the Tychonoff cube $I^m$ contains a dense Fréchet subspace X of countable pseudocharacter [3], [2, Theorem 1.5.33]. For such an X, $c(X) = \psi(X) = t(X) = \aleph_0$, and $|X|$ is great if m is. We show that any Tychonoff cube $I^m$ contains a dense Fréchet subspace which is $F_\sigma$-discrete. A space is $F_\sigma$-discrete if it is the countable union of closed discrete subspaces. Since the square of an $F_\sigma$-discrete space is $F_\sigma$-discrete and since every subset of an $F_\sigma$-discrete space is of the type $G_\delta$, any $F_\sigma$-discrete space has a $G_\delta$ diagonal. So our example answers in the negative a question of P. Simon [4], J. Ginsburg and R.G. Woods [5, question 2.5], and A. Arhangel'skii [2, problem 16]: is it true that $|X| \leq \exp \aleph_0$ for any regular space X which has the Souslin property and a $G_\delta$ diagonal. The first to solve this problem was D.B. Shachmatov. Our construction is much simpler than his and provides a space which is additionally countably tight (in fact, Fréchet).

The closed unit interval [0,1] is denoted by I. Let A be a set of indices. The points of the Tychonoff cube $I^A$ are written in the form $\{x_a : a \in A\}$. For $x \in I^A$ the set $\{a \in A : x_a \neq 0\}$ is denoted by A(x). The $\sigma$-product of the family $\{I_a : a \in A\}$ of intervals is the set $S = \{x \in I^A : A(x) \text{ is finite}\}$. The space S is Fréchet [2, Theorem 1.5.27] and has the Souslin property.

<u>Theorem</u>. Any $\sigma$-product S of closed unit intervals contains a dense subset X which is an $F_\sigma$-discrete space.

<u>Proof</u>. Choose a sequence $K_1, K_2, \dots$ of pairwise disjoint

finite subsets of I such that every nonempty open subset of I meets all but finitely many of $K_n$'s. For example, each $K_n$ may be the set of rationals of the form $(2k - 1)/2^n$, where k is a positive integer $\leq 2^{n-1}$.

For every natural n, let $S_n = \{x \in S : A(x)$ has precisely n elements$\}$. Define a subset X of S by the following rule: if a point $x \in S$ is in $S_n$, then x is in X iff $n > 0$ and all non-zero coordinates of x are in $K_n$. Clearly X is dense in S (we assume that the set A is infinite; otherwise S is a finite-dimensional cube and the theorem is obvious). We claim each $X_n = X \cap S_n$ is discrete and closed in X. For every natural n, choose a positive number $d_n$ such that $|x - y| \geq d_n$ for every two nonequal points x, y which are in the union of n sets $K_1, \ldots, K_n$. For every $x \in X$, the set $V_n(x) = \{y \in X:$ for any $a \in A(x)$, $y_a > 0$ and $|x_a - y_a| < d_n\}$ is a neighbourhood of x. Since the intersection $V_n(x) \cap X_n$ is empty for $x \in X \setminus X_n$ and equals the singleton $\{x\}$ for $x \in X_n$, it follows that each $X_n$ is closed and discrete. Hence $X = \bigcup \{X_n : n = 1,2,\ldots\}$ is $F_\sigma$-discrete.

Corollary. For any cardinal m, there exists a Tychonoff space X with the following properties: (1) X is $F_\sigma$-discrete (and therefore has a $G_\delta$ diagonal); (2) X has the Souslin property; (3) X is Fréchet; (4) $|X| > m$.

I am indebted to Professor A.V. Arhangel'skii for pointing out that the construction described here - which was intended originally to yield a space with a $G_\delta$ diagonal - yields in fact an $F_\sigma$-discrete space.

## References


[1] JUHÁSZ I.: Cardinal functions in topology - ten years later, Math. Centre Tracts 123, Amsterdam 1980.

[2] АРХАНГЕЛЬСКИЙ А.В.: Строение и классификация топологических пространств и кардинальные инварианты, Успехи матем. наук 33(1978), № 6, 29-84.

[3] АМИРДЖАНОВ Г.П.: О всюду плотных подпространствах счетного псевдохарактера и других обобщениях сепарабельности, Доклады АН СССР 234(1977), 993-996.

[4] SIMON P.: A note on cardinal invariants of square, Comment. Math. Univ. Carolinae 14(1973), 205-213.

[5] GINSBURG J., WOODS R.G.: A cardinal inequality for topological spaces involving closed discrete sets, Proc. Amer. Math. Soc. 64(1977), 357-360.



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