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CHARACTERIZATION OF THE GENERATORS OF C_0 SEMIGROUPS WHICH LEAVE A CONVEX SET INVARIANT

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Abstract: We consider the problem: Given a Banach space X, a closed convex subset $K \subseteq X$ with nonempty interior and a C_0 semigroup T_t $(t \geq 0)$ on X with generator A, find necessary and sufficient conditions for A so that $T_t K \subseteq K$ for every $t \geq 0$. To obtain a characterization of such generators we introduce two boundary principles which are generalizations of the minimum principles used in [1],[3],[8] to characterize the generators of positive C_0 semigroups on some ordered Banach spaces.

Key words: Convex set, tangent functional, linear operator, dissipative operator, C_0 semigroup, order unit space, positive semigroup.

Classification: 47B44, 47B55, 47D05, 47H20.

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I. Introduction. When given a C_0 semigroup T_t $(t \geq 0)$ on a Banach space X, an important problem is to connect the properties of T_t with those of its generator A. It is well-known, for example, that T_t is a contraction semigroup iff A is dissipative. When X is real and partially ordered by a proper wedge K, an interesting question is under what conditions on A the semigroup T_t is positive (i.e. $T_t K \subseteq K$ for every $t \geq 0$) and also when T_t is a positive contraction semigroup. This problem originates from the probability theory where positive contraction semigroups on function spaces are called Markov semigroups. Their generators were characterized by Feller ([9], see also Dynkin [7], 2.20). In [13] Phillips studied positive contraction semigroups on Banach lattices and introduced the class of the so-called dispersive operators as generators of such semigroups. They were defined in terms of an appropriate seminorm connected with the positive cone. In this setting a detailed further investigation was done in the papers [10] ,[15] .

During the last ten years,other papers appeared,this time dealing with positive C_0 semigroups on B^* algebras (see [8] ,[4] ,[3]). Some characterizations of their generators were obtained by means of positive tangent functionals.

In the two recent papers [1] , [5],the authors give a more general treatement of the subject,considering positive semigroups on real Banach spaces,partially ordered by a cone with nonempty interior.

However,it seems that the most natural formulation of the problem is this: Given a real Banach space X,a closed convex subset $K \subseteq X$ and a C_0 semigroup T_t $(t \geqq 0)$ on X with generator A,under what conditions (necessary and sufficient) on A does that semigroup leave K invariant?

When K is a convex wedge,we obtain the case of positive semigroups and when K is the unit ball we obtain the case of contraction semigroups.

Here we consider this question when K has a nonempty interior and give a characterization of A in terms of tangent functionals to K.This leads to a natural generalization of the classical maximum principle (used in the potential theory).We also introduce the weak and strong boundary principles which generalize the minimum principles used in [1],[3], [8] .

II. <u>Notations.</u> Throughout X stands for a normed real linear space with dual X'.For a convex subset $C \subseteq X$ we denote by dC its boundary,by IntC its interior and by $\bar{C}=C \cup dC$ its closure.If x belongs to dC,we denote by T(x,C) the set of tangent functionals to C at x:

(1) $T(x,C)=\{f \in X' : f \neq 0,\ f(C) \geqq f(x)\}$.

When the interior of C is nonempty,for every $x \in dC$ the set T(x,C) is nonempty according to lemma 7.2 and theorem 7.2 in [12],ch.II .

For $x \in X$ we also denote:

(2) $d(x,C)=\inf\{\|x-y\| : y \in C\}$.

We shall consider a fixed convex set $K \subseteq X$ with $\mathrm{Int}K \neq \emptyset$ and also a fixed element $e \in \mathrm{Int}\ K$. We denote by q the support function of K-e .For the properties of q we reffer to [12],ch.II,lemma 7.1

We also consider the set:

(3) $P = \{ f \in X' : f(K-e) \geqq -1 \}$.

Note that as $e \in \mathrm{Int}\, K$, $0 \in \mathrm{Int}(K-e)$ and hence q is continuous. There exists $r > 0$ such that $q(x) \leqq r\|x\|$ $(x \in X)$. It follows that P is a bounded subset of X':

If $x \in X$, $q(x/(r\|x\|)) \leqq 1$, so if $f \in P$, $-f(x) \leqq r\|x\|$. Taking also $-x$, we obtain $f(x) \leqq r\|x\|$. Hence $|f(x)| \leqq r\|x\|$, $\|f\| \leqq r$.

Therefore, according to Alaoglu's theorem (see [6], V.4.1.), the set P is compact in the topology of pointwise convergence of nets.

Throughout we denote by A a linear operator with domain $D(A) \subseteq X$ such that $e \in D(A)$. We put:

(4) $c = c(A,K,e) = \sup \{ f(Ae) : f \in P \}$

III. The boundary principles.

Definition 1. We say that the operator A satisfies the weak boundary principle (w.b.p.) with respect to K, iff for every x in $D(A) \cap dK$ there exists $f \in T(x,K)$ such that $f(Ax) \geqq 0$.

Now we prove:

Theorem 1. The following conditions are equivalent:

(5) The operator A satisfies the w.b.p. with respect to K;

(6) For every $x \in D(A)$, $x \overline{\in} \bar{K} = K \cup dK$, there exists $f \in P$ with $f(e-x) = q(x-e)$ and $f(Ax) + (q(x-e)-1)f(Ae) \geqq 0$;

(7) There exists $w \geqq c(A,K,e)$ such that for every $x \in D(A)$, $x \overline{\in} \bar{K}$, there is a $f \in P$ with $f(e-x) = q(x-e)$ and $f(Ax) + (q(x-e)-1)w \geqq 0$;

(8) If $t > 0$, $tc(A,K,e) < 1$, $x \in D(A)$ and $x - tAx \in \bar{K}$, then $x \in \bar{K}$;

(9) There exists $w \geqq c(A,K,e)$ such that if $t > 0$, $tw < 1$, $x \in D(A)$ and $x - tAx \in \bar{K}$, then $x \in \bar{K}$;

Proof: (5) $\to$ (6): Let $x \in D(A) \cap (X \setminus \bar{K})$. Then $x - e \overline{\in} \overline{K-e}$ and hence $q(x-e) > 1$. Put $a = q(x-e)$. As $q((x-e)/a) = 1$, $(x-e)/a$ is in $d(K-e) = dK - e$ and $y = (x-e)/a + e \in dK$. According to (5) there exists $f \in T(y,K)$, $f(Ay) \geqq 0$, i.e. $f(Ax) + (a-1)f(Ae) \geqq 0$. For f we also have: $f(K) \geqq f(e) + f(x-e)/a$. As $e \in \mathrm{Int}\, K$, we see that $f(e-x) > 0$. Multiplying f by a positive number, we can assume that $f(e-x) = a$. Hence $f(K-e) \geqq -1$, $f \in P$. (6) $\to$ (7) and (8) $\to$ (9) are direct, putting $w = c(A,K,e)$.

Now (7) $\rightarrow$ (9): Let $x \in D(A), t > 0, tw < 1$ (w - as in (7)) and $x-tAx \in \overline{K}$. Suppose $x \notin \overline{K}$ and let f be as in (7). We have: $f(x-tAx-e) \geqq -1$, hence $-tf(Ax) \geqq -1 + f(e-x) = a-1$ ($a=q(x-e) > 1$). From (7): $-tf(Ax) \leqq (a-1)tw$, so $tw \geqq 1$ - a contradiction. In the same way (6) implies (8), as $f(Ae) \leqq c(A,K,e)$ ($f \in P$).

It remains to prove (9) $\rightarrow$ (5). First we fix $t > 0$, $tw < 1$ (w - as in (9)). Let $x \in D(A) \cap dK$. Then $x-e \in d(K-e)$ and $q(x-e)=1$. For every $s > 1$, $s(x-e) \notin \overline{K}-e$ as $q(s(x-e)) > 1$. Hence $s(x-e)+e$ is not in $\overline{K}$ and (9) implies that $y_s = s(x-e) - stA(x-e) - tAe$ is not in $\overline{K}-e$. According to lemma 7.2 in [12], ch.II, or theorem 0.2.4. in [11], there exists a nonzero $f_s \in X'$ separating $\overline{K}-e$ and y_s. For f_s we have $f_s(\overline{K}-e) \geqq f_s(y_s)$ and as $0 \in \mathrm{Int}(K-e)$, we obtain $f_s(y_s) < 0$. We can assume that $f_s(y_s) = -1$ and so $f_s \in P$. As P is compact in the topology of pointwise convergence of nets, the net f_s with s in the downwards directed set $(1, \infty)$ has a convergent subnet with limit $f_t \in P$. For convenience we assume that $f_s \rightarrow f_t$ poinwise as $s \rightarrow 1+$. We have $y_s \rightarrow x-e-tAx$ ($s \rightarrow 1+$) and therefore:

$f_t(\overline{K}-e) \geqq -1 = f_t(x-e)-tf_t(Ax)$, $f_t(\overline{K}) \geqq f_t(x)-tf_t(Ax)$ and as $x \in \overline{K}$,

$f_t(Ax) \geqq 0$.

Now we let $t \rightarrow 0+$ (in the downwards directed set of positive real numbers) and assume as above that $f_t \rightarrow f \in P$. We obtain: $f(K) \geqq f(x)$, $f(x-e)=-1$, hence $f \neq 0, f \in T(x,K)$. From above we also obtain $f(Ax) \geqq 0$. The proof is completed.

We shall introduce also a strong boundary principle (s.b.p.).

<u>Definition 2.</u> We say that A satisfies the <u>strong boundary principle</u> with respect to K, iff for every $x \in dK \cap D(A)$ and every $f \in T(x,K)$ we have $f(Ax) \geqq 0$.

As $\mathrm{Int}\, K \neq \emptyset$, $T(x,K) \neq \emptyset$ for every $x \in dK$ and hence the s.b. p. "implies" the w.b.p.. We have also the following criteria:

<u>Lemma 1.</u> The following are equivalent:

(10) A satisfies the s.b.p. with respect to K;

(11) For every $x \in D(A) \cap (X \setminus \overline{K})$ and every $f \in P$ with $f(e-x)=q(x-e)$ we have $f(Ax) + (q(x-e)-1)f(Ae) \geqq 0$.

<u>Proof:</u> (10) $\rightarrow$ (11): Let $x \in D(A) \cap (X \setminus \overline{K})$ and $f \in P, f(e-x) = q(x-e) = a > 1$. Then $y=(x-e)/a+e \in dK \cap D(A)$ and from $f(K-e) \geqq -1$ we obtain $f(K) \geqq f(e)-1 = f(e)-q((x-e)/a) = f(y)$. Hence $f \in T(y,K)$.

According to (10) we have $f(Ay) \geqq 0$, i.e. $f(Ax)+(s-1)f(Ae) \geqq 0$.

Now (11) $\Rightarrow$ (10): Let $x \in dK \cap D(A)$ and $f \in T(x,K)$. Then $x-e$ is in $d(K-e)$, hence $q(x-e)=1$. For $s > 1$, $q(s(x-e)) > 1$, so $s(x-e)$ is not in $K-e$ and $y=s(x-e)+e \in (X \setminus \bar{K}) \cap D(A)$. Let $f \in P$, $f(e-x)=q(x-e)=1$ be as in (11). Then $f(K) \geqq f(e)-f(e-x)=f(x)$, hence $f \in T(x,K)$. From (11) we also have $f(Ay)+(s-1)f(Ae) \geqq 0$, which implies $f(Ax) \geqq 0$.

Lemma 2. Let X be complete and K be closed. Then A satisfies the s.b.p. with respect to K if and only if:

(12) $\lim_{t \to 0+} d(x+tAx,K)/t = 0$ for every $x \in dK \cap D(A)$.

This is a straightforward corollary from lemma 7.3, ch.VI in [12]. See also [17]. (Note that our definition of tangent functionals differs from that on p.53 in [12] in the direction of the inequality. We want $f \in T(x,K)$ to be positive when K is a cone.)

IV. C_0 semigroups leaving K invariant. In this section we assume that X is complete and K is closed. Let $T_t (t \geqq 0)$ be a C_0 semigroup of operators on X (see [6], VIII.1.) and let A be its infinitesimal generator. In this case $D(A)$ is dense in X and as Int $K \neq \emptyset$, the assumption $e \in D(A) \cap \text{Int } K$ is no loss of generality. Remind that the resolvent $(I-tA)^{-1}$ exists as a bounded operator on X and maps it onto $D(A)$ for all sufficiently small positive t .

In [17] Martin considered the condition (5) and proved that it (and hence (10), (12)) implies the invariance of K for $(I-tA)^{-1}$ (for all $t > 0$ sufficiently small) and hence for T_t $(t \geqq 0)$. He proved this for general evolution systems. In the case of C_0 semigroups we complement his result in the following theorem.

Theorem 2. The conditions (5), (6), (7), (10), (11), (12) and (13), (14) (see below) are equivalent

(13) $(I-tA)^{-1}K \subseteq K$ for all $t > 0$ sufficiently small ;

(14) $T_t K \subseteq K$ for all $t \geqq 0$.

If the condition:

(15) For every $x \in D(A) \cap dK$ there exists $f \in T(x,K)$ with $f(Ax) > 0$ holds, then we have:

(16) $T_t x \in \text{Int } K$ for all $t \geqq 0$ and all $x \in \text{Int } K$ such that $T_s x$ is in $D(A)$ when $s > 0$.

In particular:

(17) $T_t(D(A)\cap \mathrm{Int}\ K) \subseteq D(A)\cap \mathrm{Int}\ K$ for all $t \geqq 0$.

Finally, if (5) holds and $A(e)=0$, then:

(18) $T_t K_s \subseteq K_s$ for every $t \geqq 0$, where $0 \leqq s \leqq 1$ and $K_s = sK+(1-s)e$.

<u>Proof.</u> Having in mind theorem 1 and lemmas 1 and 2, we see that for the first part of the theorem we need only show (13)$\to$ (14) $\to$ (10). According to the well-known representation:

$T_t x = \lim_{n\to\infty} (I-(t/n)A)^{-n}x$ for every $x \in X$ and $t \geqq 0$, (13) implies (14). Suppose now that $x \in D(A)\cap dK$ and $f \in T(x,K)$. If (14) holds, $T_t x \in K$ $(t \geqq 0)$ and therefore $f(T_t x) \geqq f(x)$ which implies:

$f(Ax)=f(\lim_{t\to 0+} (T_t x-x)/t) \geqq 0$.

Let now (15) hold and $x \in \mathrm{Int}\ K$ be such that $T_s x \in D(A)$ for all $s > 0$. Suppose $T_t x \in dK$ for some $t > 0$. We may assume that t is the smallest positive number with this property (i.e. $T_s x \in \mathrm{Int}\ K$ when $0 \leqq s < t$). Let $f \in T(T_t x,K)$ with $f(AT_t x) > 0$. For the differentiable real function $h(s)=f(T_s x)$ $(0 \leqq s \leqq t)$ we have $h(s) > h(t)$ when $0 \leqq s < t$. Hence $h'(t)=f(AT_t x) \leqq 0$ - a contradiction.

Finally, let A satisfy the w.b.p. with respect to K and $A(e)=0$. Then it is easy to see that A satisfies the w.b.p. with respect to K_s for every s between 0 and 1. Really, let $s > 0, s < 1$ and let $x \in dK_s = sdK+(1-s)e$. Then $y_s=(x-(1-s)e)/s \in dK$ and there exists $f \in T(y_s,K)$ with $f(Ay_s) \geqq 0$ (when $x \in D(A)$; then $y_s \in D(A)$ too as $D(A)$ is a linear subspace). It follows that $f \in T(x,K_s)$ and $f(Ax) \geqq 0$ as $Ax=Ay_s/s$.

The proof is completed.

Remark. If (15) holds, then obviously $0 \bar{\in} dK$. When $0 \in dK$ conditions (15) and (16) can be modified so that the implication (15) $\to$ (16) to hold again. Considerations are left to the reader.

V. <u>Dissipative operators.</u> Let X be real and normed, let K be the unit ball in X: $K=\{x\in X : \|x\| \leqq 1\}$ and let $e=0$. Then for every $x \in X$, $q(x)=\|x\|$. The condition (6) takes the form:

(6') For every x $D(A)$ with $\|x\| > 1$ there exists $f \in X'$ such that $f(K) \geqq -1, f(x)= -\|x\|$ (hence $\|f\|=1$) and $f(Ax) \geqq 0$.

As A is linear, the condition $\|x\| > 1$ can be replaced by just $x\neq 0$ and taking $-x$ instead of x we obtain the above condition in the form:

(6'') For every $x \in D(A), x \neq 0$ there exists $f \in X'$ with $\|f\| = 1$, $f(x) = \|x\|$ and $f(Ax) \leqq 0$.

This is the well-known definition of dissipative operators.With x, f as in (6'') we have:

(21) $\|x\| = f(x) \leqq f(x) - tf(Ax) = f(x-tAx) \leqq \|x-tAx\|$ for every $t \geqq 0$.

Conversely, (21) implies (6'') according to theorem 9.5,ch.V in [6] or theorem 5.1,ch.II in [12] .

In this case theorem 2 represents the well-known result (see [6], ch.VIII,corollary 1.14) that a C_0 semigroup T_t ($t \geqq 0$) on a real Banach space is a contraction semigroup iff its generator A is dissipative.

VI. The case of order unit space. In this section we use the terminology of [11].The setting is similar to that in [1],[5].

Let X be a partially ordered real linear space with proper cone K of positive elements.Let e be an order unit and $\|.\|$ - the order unit seminorm.We assume that $\|.\|$ is a norm (this is so,iff the ordering is almost Archimedean - see [11],p.12 and p. 116) and that K is lineally closed (this is so,iff the ordering is Archimedean - see [11],1.1.4,p.13). In this case $e \in \mathrm{Int}\, K$.As usual,we write $x \geqq y$ (or $y \leqq x$) iff $x-y \in K$. The set P (see (3)) consists of all positive linear functionals f on X with $f(e) \leqq 1$ (proof: as K is a cone,if $f \in P$, $tf(K) \geqq -1+f(e)$ for every $t > 0$,hence f is positive and as $0 \in K$, $f(e) \leqq 1$;every positive functional is bounded - see theorem 3.7.2,p.118,[11],so the converse follows).

The support function q of K-e is given by:

(22) $q(x) = \inf\{ t > 0 : te+x \geqq 0 \}$ $(x \in X)$.

And $x \geqq 0$ iff $q(x)=0$.

It is easy to see that if $x \in dK$,the set $T(x,K)$ consists of all positive linear functionals $f \neq 0$ with $f(x)=0$: If $f \in T(x,K)$,we have $f(K) \geqq f(x)/t$ for every $t > 0$,hence f is positive,as x and 0 are in K, $0 \geqq f(x) \geqq 0$,hence $f(x)=0$.The converse is trivial.Note also that if $f \in P, f \neq 0$,then $\|f\| = f(e) \neq 0$ (see 3.7.2.,p.118,[11]).It follows easily that dK consists of all $x \in K$ for which there exists $f \in P, f \neq 0$,with $f(x)=0$.

We denote for every $x \in X$:

(23) $p(x) = \inf\{ t \in R : te-x \geqq 0 \}$ (R - the reals) .

In this setting from theorem 1 we obtain:

Corollary 1. For the operator A the following conditions are equivalent:

(24) If $x \in D(A) \cap dK$, there exists $f \in P, f \neq 0, f(x)=0$ and $f(Ax) \geqq 0$;

(25) If $x \in D(A)$ and $p(x) > 0$ there exists $f \in P, f(e)=1, f(x)=p(x)$ and $f(Ax) \leqq f(Ae)p(x)$;

(26) If $x \in D(A)$ and $q(x) > 0$, there exists $f \in P$, $f(e)=1$, $f(x)= -q(x)$ and $f(Ax)+q(x)f(Ae) \geqq 0$;

(27) There exists $w \geqq c(A,K,e)$ such that if $x \in D(A), q(x) > 0$, there exists $f \in P, f(e)=1, f(x)= -q(x)$ and $f(Ax)+q(x)w \geqq 0$;

(28) If $t > 0, tc(A,K,e) < 1, x \in D(A)$ and $x-tAx \geqq 0$, then $x \geqq 0$;

(29) There exists $w \geqq c(A,K,e)$ such that if $t > 0, tw < 1, x \in D(A)$ and $x-tAx \geqq 0$, then $x \geqq 0$.

Proof. First we show that the condition (6) of theorem 1 takes the form (26). Let (26) hold and let $x \in D(A)$, $x \overline{\in} \bar{K}=K$. Then $q(x) > 0$. Let f be as in (26). From $f(x)= -q(x)$ and $f(e)=1$ we obtain $f(e-x)=q(x-e) > 1$ as $q(x-e)=q(x)+1$ (a direct verification) and (6) follows. Conversely, let (6) hold and let $x \in D(A), q(x) > 0$. Then $q(x-e)=q(x)+1=a > 1$ and hence $x \overline{\in} \bar{K}$. Let f be as in (6). Then $f \in P$ and $f(e-x)=q(x-e)=a$. We have $(x-e)/a+e \in dK$ and as f is tangent to K at that point (straightforward), $f((x-e)/a+e)=0$. Hence $f(x)+q(x)f(e)=0$. As $f \neq 0, f(e) > 0$ and dividing f by $f(e)$ we obtain the necessary functional.

It remains to show that (25) and (26) are equivalent. This follows from the observation that if $p(x) > 0$, then $p(x)=q(-x)$ and if $q(x) > 0$, then $q(x)=p(-x)$. The proof is completed.

We included (25) in the above set of conditions with a definite aim. It is a direct generalization of the well-known weak maximum principle, upon which we shall comment in the next section.

Sometimes it is convenient to consider another conditions equivalent to those in corollary 1.

Proposition 1. The conditions (24) - (29) are equivalent to (cf. [5]):

(30) $q(x-tAx) \geqq (1-tc)q(x)$ for every $x \in D(A)$ and every $t > 0$, $tc < 1$ $(c=c(A,K,e))$;

(31) There exists $w \geqq c(A,K,a)$ such that:

$$q(x-tAx) \geqq (1-tw)q(x) \quad \text{for every } x \in D(A), t > 0, tw < 1 .$$

<u>Proof:</u> Let $x \in D(A)$ and $t > 0, tc < 1$. Let $q(x) > 0$ and f be as in (26) (if $q(x)=0$, (30) holds). Then:
$t(f(Ax)+q(x)c)=f(tAx-x+(1-tc)x) \geqq 0$. We also have $-f(y) \leqq q(y)$ (as $q(y)e+y \geqq 0$) for every $y \in X$. With $y=x-tAx$ (30) follows. In the same way (31) follows from (27). Conversely, (30) and (31) obviously imply (28) and (29) respectively.

Let $a=\max(c,0)$ or $a=\max(w,0)$ (w - as in (31)). Then for $t > 0$, $ta < 1$ we put $s=t/(1-ta)$ and (30),(31) take the form:

(32) $\quad q(x-s(A-aI)x) \geqq q(x)$ for every $x \in D(A)$ and every $s \geqq 0$;

which means that $A-aI$ is q-dissipative (cf. [1]).

<u>Corollary 2.</u> Let X be complete and let T_t $(t \geqq 0)$ be a C_0 semigroup of operators on X with generator A. Then each of the conditions (24) - (32) implies the positivity of T_t and vice-versa. The form of the strong boundary principle, the form of the conditions (11) and (12) in this case and other details are left to the reader.

VII. <u>The maximum principles.</u> Let $C(M)$ be the real Banach space (with the "sup" norm) of all real continuous functions on a compact topological space M. Let A ($D(A) \subseteq C(M)$) be a linear operator. Consider the condition:

(33) For every $u \in M$ and every $x \in D(A)$ such that:
$x(u)=\sup\{x(v):\ v \in M\} \geqq 0$, we have $Ax(u) \leqq 0$.

If A satisfies (33), then its resolvent $(I-tA)^{-1}$ is defined on $(I-tA)D(A)$ and is a positive contraction operator there (with respect to the usual order: $x \geqq 0$ when $x(u) \geqq 0$ for $u \in M$) for every $t > 0$. If A is the generator of a C_0 semigroup $T_t (t \geqq 0)$ on $C(M)$, the same follows for T_t and vice-versa (see Dynkin [7], 2.20). It was noticed that this is true also when A satisfies:

(34) For every $x \in D(A)$ such that $l=\sup\{x(v):\ v \in M\} > 0$ there exists $u \in M$ with $x(u)=l$ and $Ax(u) \leqq 0$.

These conditions can be considered when M is locally compact and $D(A) \subseteq C^{o}(M)$ (the bounded real continuous functions on M with the "sup" norm, which are zero at infinity), as is done in the potential theory, where (33) is known as the (strong) maximum prin-

ciple and (34) as the weak maximum principle (see [14] ,[21]).
In [4] the strong maximum principle was generalized for operators on non-unital B^*-algebras and in [3] the weak maximum principle was generalized for operators on B^*-algebras with a unit.
In the case of operators on ordered linear spaces, the condition (25) in the preceding section may be considered as a generalization of (34). Further generalizations are (6),(7) and (11).

VIII. <u>Some additional remarks.</u> Let X be complete and let A ($D(A) \subseteq X$) be a single-valued non-linear dissipative operator (with values in X) in the sense of [2] ,ch.II,§ 3, such that:
$\overline{D(A)} \subseteq (I-tA)D(A)$ for every $t > 0$. Then the limit:

$$S_t x = \lim_{n \to \infty} (I-(t/n)A)^{-n}x \quad \text{exists for every } x \in \overline{D(A)} \text{ and } t > 0,$$

and is a contraction semigroup on $\overline{D(A)}$ (in the sense of [2],ch.III, 1.1 ;see also theorem 1.3 on p. 104 there).

<u>Proposition 2.</u> Let K be a closed convex subset of D(A) with a non-empty interior and let $e \in D(A) \cap \operatorname{Int} K$. Then each of the conditions (6),(7),(11) implies $S_t K \subseteq K$ for all $t \geqq 0$.
The proof follows from the observation that (6) implies (8) (or (7) implies (9)) in theorem 1 without using the linearity of A.
The conditions (6),(7),(11) can be modified for multilinear operators and the above proposition can be generalized for such operators in an obvious way (via theorem 1.3 on p.104 in [2]).
If X,K,e are as in section VI, $\overline{D(A)}=X$ and X is complete, the condition (26) (or (27)) in corollary 1 implies the positivity of S_t ($t \geqq 0$), and if A is odd (i.e. $A(-x)=-Ax$ for $x \in D(A)$), the same follows from (25).
The question when a closed set K is invariant for a given (nonlinear) semigroup (evolution system, flow) was studied by many authors. The most often used condition for the generator which implies the invariance is (12) (with the necessary modifications for flows). This condition originates from Nagumo [19] . The progress in this subject can be traced in [12] ,ch.VI;[16],§5; and also [17] , [18],[20],[22],[23] . See also the references there.

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