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A NOTE ON CONTINUITY PRINCIPLE IN POTENTIAL THEORY

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Abstract: In this note a proof is given of a continuity property of Evans-Vasilesco type for general potentials of signed measures.

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Let X be a locally compact Hausdorff topological space and let K be a continuous function-kernel on X, i.e. an extended-real-valued positive continuous (in the wide sense) function on $X \times X$ which is finite off the diagonal $\Delta = \{[x,x]; x \in X\}$ and strictly positive on Δ. Given a Radon measure $\mu \geq 0$ on X we denote by

$$K\mu \quad : \quad x \mapsto \int_X K(x,y)\,d\mu(y)$$

its potential. Let us recall that K is termed regular (cf. [4]) if it satisfies the following continuity principle:

(C) If $\mu \geq 0$ is a Radon measure with a compact support spt μ such that the restriction of $K\mu$ to spt μ is finite and continuous, then $K\mu$ is necessarily finite and continuous on the whole space X.

In applications one often has to consider potentials of signed measures; given a signed Radon measure ν with the Jordan decomposition $\nu = \nu^+ - \nu^-$, then its potential is defined as $K\nu = K\nu^+ - K\nu^-$ provided the difference is meaningful everywhere on X. Because of possible "cancellation of discontinuities" it may happen that $K\nu$ is finite and continuous even though $K\nu^+$, $K\nu^-$ are discontinuous (cf. [1],[10]). Thus the classical Evans-Vasilesco theorem does not permit the conclusion that a Newtonian potential of a signed measure ν must be continuous everywhere provided its restriction to spt ν is continuous. In a discussion on the occasion of the conference " 5.Tagung über Probleme und Methoden der Mathematischen Physik " (held in Karl-Marx-Stadt in May 1973) B. W. Schulze raised the question of validity of the extended Evans-Vasilesco theorem for Newtonian potentials of signed measures. Using refined tools of abstract potential theory I. Netuka was able to supply in [10] a proof of the corresponding result valid for potentials on harmonic spaces satisfying the strong domination axiom (cf. [5]). It is the purpose of this note to give an elementary proof of a related continuity property of signed potentials for kernels K obeying the following domination principle:

(D) If $\mu_1 \geq 0$ and $\mu_2 \geq 0$ are compactly supported Radon measures with finite potentials such that $K\mu_1 \leq K\mu_2$ on spt μ_1, then $K\mu_1 \leq K\mu_2$ on the whole space X.

<u>Remark.</u> The classical Riesz kernel $[x,y] \mapsto |x-y|^{\alpha-n}$ on the Euclidean space $X = R^n$ satisfies (D) provided $0 < \alpha \leq 2 < n$ (cf. [11], [7] and Theorem 1.29 in [9]).

The reader is referred to [6], [7], [12] for general investigation of potential kernels on locally compact spaces.

The following result was presented by the author in the Analysis Seminar (held in Prague in October 1975; the proof has been included in [8], p. 245).

Theorem 1. Let K be a strictly positive continuous function-kernel satisfying (D) and suppose that ν is a compactly supported signed Radon measure with a finite potential $K\nu$. If the restriction of $K\nu$ to spt ν is upper semicontinuous, then $K\nu$ is upper semicontinuous on the whole space.

The proof is based on the following two known simple lemmas.

Lemma 1. Any continuous function-kernel K enjoying (D) is regular.

Proof. Cf. [7], Corollary 1.3.10 and proof of Proposition 1.3.8.

Lemma 2. If K is regular and μ is a compactly supported Radon measure such that $K\mu$ is finite on spt μ, then there exists an increasing sequence of Radon measures $\mu_n \leq \mu$ such that the potentials $K\mu_n$ are finite and continuous on X and converge pointwise (as $n \uparrow \infty$) to $K\mu$ on X.

Proof. Cf. Proposition 4 in Chap. II in [3] or Lemma 1.2.4 in [7].

Proof of Theorem 1. If ν^+ is trivial, then $K\nu = -K\nu^-$ is upper semicontinuous on X. Assume $\nu^+(X) > 0$, fix $z \in X$ and $\varepsilon > 0$. Lemma 2 guarantees the existence of an increasing sequence of Radon measures $\mu_n \leq \nu^+$ with finite continuous potentials such that

$$(1) \qquad 0 < K\mu_n \uparrow \quad K\nu^+ \quad \text{as} \quad n \uparrow \infty$$

as well as the existence of a Radon measure μ with a continuous potential such that

$$(2) \qquad \mu \leq \nu^- \quad , \quad K(\nu^- - \mu)(z) < \varepsilon K\mu_1(z) .$$

Consequently,

$$(3) \qquad K(\nu + \mu - \mu_n) \downarrow \quad -K(\nu^- - \mu) \leq 0 < \varepsilon K\mu_1$$

and upper semicontinuity of the restriction of $K\nu$ to spt ν implies that also the restrictions of $K(\nu + \mu - \mu_n)$ to spt ν are upper semicontinuous. In view of (3), for n large enough $K(\nu + \mu - \mu_n) \leq \varepsilon K\mu_1$ on spt ν or , which is the same,

$$(4) \qquad K(\nu^+ + \mu) \leq \varepsilon K\mu_1 + K\mu_n + K\nu^- .$$

Noting that spt $(\nu^+ + \mu) \subset$ spt ν we conclude by (D) that (4) holds everywhere on X . We have by (2),(1)

$$- K\mu(z) < \varepsilon K\mu_1(z) - K\nu^-(z) ,$$

$$K\mu_n(z) \leq K\nu^+(z) .$$

Hence we get for $f = \varepsilon K\mu_1 - K\mu + K\mu_n$

$$f(z) < K\nu(z) + 2\varepsilon K\mu_1(z) .$$

Since f is continuous, there is a neighbourhood V of z such that

$$x \in V \Rightarrow f(x) < K\nu(z) + 2\varepsilon K\mu_1(z)$$

which together with (4) gives

$$x \in V \Rightarrow K\nu(x) < K\nu(z) + 2\varepsilon K\nu^+(z)$$

and the upper semicontinuity of $K\nu$ at z is established.

Remark. The above theorem may fail to hold for regular kernels not fulfilling (D) (cf. example 9 in [8], pp.246-248).

R. Wittmann (cf. [13]) has recently proposed a new approach to continuity properties of signed potentials which avoids kernels and works in the framework of cones of functions. His scheme may be desribed as follows:

Let* X be a locally compact Hausdorff topological space and P a convex cone of non-negative continuous functions on X containing a strictly positive function. Denote by S the convex cone of all (finite) functions which are pointwise limits of increasing sequences in P. Let $Q \subset X$ be a compact set and suppose that $P_Q \subset P$ is a convex cone posessing the following property:

(D_Q) $(p \in P_Q,\ q \in P,\ p \leq q \text{ on } Q) \Rightarrow p \leq q \text{ on } X$.

(Clearly, (D_Q) implies the same property with any $q \in S$.) Denote by P_Q^* the linear space of all functions f on X for which there exist sequences $\{p_n\}$, $\{q_n\}$ in P_Q and an $s \in S$ such that

(i) $|p_n - q_n| \leq s \quad (n \in N)$,

(ii) $\lim_{n\to\infty} (p_n - q_n)(x) = f(x), \quad x \in X$.

Then the following Wittmann's theorem holds:

<u>Theorem 2.</u> Any $f \in P_Q^*$ is already continuous throughout X if only its restriction to Q is continuous.

This theorem can be used to get the following corollary of Theorem 1:

If $K\nu$ is a finite non-trivial compactly supported signed potential whose restriction to spt $\nu = Q$ is continuous, then $K\nu$ is continuous on the whole space.

We denote by P the cone of all finite continuous potentials $K\mu$ of compactly supported Radon measures $\mu \geq 0$ and by P_Q the cone of all $K\mu \in P$ with spt $\mu \subset Q$. Clearly, (D) implies (D_Q). By Lemma 2 there are sequences $p_n \in P_Q$, $q_n \in P_Q$ with $p_n \uparrow K\nu^+$, $q_n \uparrow K\nu^-$, so that $|p_n - q_n| \leq K(\nu^+ + \nu^-) \in S$. Theorem 2 then implies continuity of $K\nu$ on X.

R.Wittmann's proof of Theorem 2 is based on an application of the Hahn-Banach theorem as employed by H.Bauer in [2]. It is perhaps of interest to note that the direct approximation technique used for the proof of Theorem 1 above may also be used to provide the following alternative of the proof of Wittmann's theorem.

<u>Proof.</u> Let f be given by (ii), where p_n, $q_n \in P_Q$ enjoy (i) for suitable $s \in S$; we may clearly suppose that s is strictly positive on X. Let us equip the space of continuous functions g on Q with the norm

$$\|g\|_s = \inf \{ \lambda \geq 0 ; |g| \leq \lambda s \text{ on } Q \} .$$

The resulting normed space $C_s(Q)$ has dual $C_s^*(Q)$ which is represented by those signed Radon measures $\nu = \nu^+ - \nu^-$ on Q, for which s is $(\nu^+ + \nu^-)$ - integrable over Q. The conditions (i), (ii) mean that the sequence $\{p_n - q_n\}_{n=1}^{\infty}$ converges weakly to f in $C_s(Q)$. Consequently, there is a sequence $\{u_n^1\}_{n=1}^{\infty}$ formed by finite convex combinations of the elements $(p_n - q_n)$ which converges to f in $C_s(Q)$; we may thus assume that $\|u_n^1 - f\|_s < 2^{-3}$ $(n \in N)$. Applying the same reasoning to the sequence

$$\{p_n - q_n\}_{n=k}^{\infty} \tag{5}$$

we get for any $k \in N$ a sequence $\{u_n^k\}_{n=1}^{\infty}$ of convex combinations of elements of (5) which converges to f in $C_s(Q)$ and satisfies

$$\|u_n^k - f\|_s < 2^{-k-2} , \; n \in N . \tag{6}$$

Put $u_n = u_n^n$, $n \in N$. The sequence $\{u_n\}_{n=1}^{\infty}$ converges to f

pointwise on X, because u_k is a convex combination of elements of (5) and (ii) holds. It follows from (6) that $\|u_n - u_{n+1}\|_s < 2^{-n-1}$ whence, in view of the definition of the norm $\|\ldots\|_s$,

$$(7) \qquad u_n - 2^{-n} s \uparrow f \ , \ u_n + 2^{-n} s \downarrow f \quad (n \uparrow \infty)$$

on Q. Since $u_n = p_n^* - q_n^*$ for suitable p_n^*, $q_n^* \in P_Q$, (D_Q) implies that the sequence $\{u_n - 2^{-n}s\}$ is nondecreasing on X and the sequence $\{u_n + 2^{-n}s\}$ is nonincreasing on X, so that (7) holds on X. Note that, for any $p \in P_Q$ and $\sigma \in S$ the following implication is true:

$$(8) \qquad f \leq \sigma - p \quad \text{on } Q \Rightarrow f \leq \sigma - p \quad \text{on } X .$$

Indeed, the inequality $u_n - 2^{-n}s \leq \sigma - p$ can be rewritten in the form $p_n^* + p \leq \sigma + 2^{-n}s + q_n^*$ which, according to (D_Q), holds on X whenever it holds on Q. Using (7) one gets (8). Let now z be an arbitrarily fixed point of X. We have by (7)

$$u_n(z) < f(z) + 2^{-n+1}s(z) ,$$

whence we conclude by continuity of u_n that for suitable neighbourhood V_n of z

$$(9) \qquad x \in V_n \Rightarrow u_n(x) < f(z) + 2^{-n+1}s(z) .$$

There is a sequence $r_k \in P$ such that $r_k \uparrow s$ $(k \uparrow \infty)$. Note that

$$f < u_n + 2^{-n+1}s$$

on Q by (7). Since the restriction of f to Q is continuous, for sufficiently large k_n

$$f < u_n + 2^{-n+1}r_{k_n}$$

on Q, whence by (8)

$$f \leq u_n + 2^{-n+1} r_{k_n} \quad \text{on } X .$$

We have thus by (9)

$$x \in V_n \Longrightarrow f(x) \leq f(z) + 2^{-n+1} s(z) + 2^{-n+1} r_{k_n}(x) ,$$

$$\limsup_{x \to z} f(x) \leq f(z) + 2^{-n+1} s(z) + 2^{-n+1} r_{k_n}(z) \leq$$

$$\leq f(z) + 2^{-n+2} s(z)$$

for any $n \in N$. This proves that f is upper semicontinuous at z .

Remark. Note that local compactness of X was not needed in the above proof.

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