COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
25,1 (1984)

ON REGULAR RING-SEMIGROUPS AND SEMIRINGS

J. ZELEZNIKOW

Abstract: Regular and orthodox ring-semigroups and semirings are characterized, as well as ring-semigroups with chain conditions on idempotents and principal ideals. Congruences on additively regular semirings are also considered.

Key words: Ring-semigroup, additively inverse semiring, orthodox semigroup, congruence, Green's relations.

Classification: Primary 16A78
Secondary 20M75, 16A30

---

1. Introduction: In a semigroup $(S,\cdot)$ we put $E = \{e \in S : e^2 = e\}$ and $V(x) = \{a \in S : x\cdot a\cdot x = x$ and $a\cdot x\, a = a\}$ for all $x \in S$. If $V(x) \neq \square$, then the element x is said to be regular. If each element of S is regular, then the semigroup S is said to be regular. If S is a regular semigroup, and E is a subsemigroup of S, then S will be said to be an orthodox semigroup. A regular semigroup in which $e\cdot f = f\cdot e$ for all $e,f \in E$, is said to be an inverse semigroup.

We use the definitions and notation of [1].

A semigroup $(S,\cdot)$ is a ring-semigroup if there exists a binary operation $+ : S \times S \to S$ such that $(S,+,\cdot)$ is a ring.

In [12], the structure of orthodox ring-semigroups was considered. Such semigroups are inverse. In the proof of this theorem, the concept of an additivelv inverse semiring is required.

Definition 1: A triple $(S,+,\cdot)$ is a semiring if S is a set, and $+,\cdot$ are binary operations satisfying

(i) $(S,+)$ is a semigroup,

(ii) $(S,\cdot)$ is a semigroup,

(iii) $a\cdot(b + c) = a\cdot b + a\cdot c$, $(a + b)\cdot c = a\cdot c + b\cdot c$, for all $a,b,c \in S$.

Definition 2: A semiring $(S,+,\cdot)$ is said to be an additively inverse semiring if $(S,+)$ is an inverse semigroup.

The following theorem of Karvellas allows us to prove many results for additively inverse semirings.

Result 3: ([7] Theorem 7.)

In an additively inverse semiring $(S,+,\cdot)$, if $a \in aS \cap Sa$ for all $a \in S$, then S is additively commutative (and hence a semilattice of groups).

In a semiring $(S,+,\cdot)$ we put $E^{[+]} = \{x \in S : x + x = x\}$ and $E^{[\cdot]} = \{e \in S : e\cdot e = e\}$.

2. Regular ring-semigroups: We can now prove:

Result 4: ([12] Theorem 9.)

Let $(S,+,\cdot)$ be any additively inverse semiring in which $(S,\cdot)$ is regular. Then the following conditions are equivalent:

(i) $\forall e,f \in E^{[\cdot]}$, $(e\cdot f = 0 \Rightarrow f\cdot e = 0)$.

(ii) $\forall e \in E^{[\cdot]}$, $\forall x \in S$, $(e\cdot x = 0 \Rightarrow x\cdot e = 0)$.

(iii) $\forall n \in \mathbb{N}$, $\forall x \in S$, $(x^n = 0 \Rightarrow x = 0)$.

(iv) $\forall x \in S$, $(x^2 = 0 \Rightarrow x = 0)$.

(v) $\forall x,y \in S$, $(x\cdot y = 0 \Rightarrow y\cdot x = 0)$.

Further, each is implied by

(vi) $(S,\cdot)$ is orthodox.

Example 5: In an arbitrary regular semigroup $(S,\cdot)$, condition (i) of Theorem 4 does not imply condition (ii), and $(S,\cdot)$ being orthodox does not imply condition (ii). To see this we may take any Brandt semigroup $S = \mathcal{M}^0(G,I,I,\Delta)$ in which $|I| \geq 2$.

Thus this semigroup cannot be the multiplicative semigroup of an additively inverse semiring.

Result 6: ([12] Theorem 13.)

In a regular ring-semigroup $(S,\cdot)$ the following conditions are equivalent:

(i) $(S,\cdot)$ is orthodox.

(ii) $\forall e,f \in E$, $(e\cdot f = 0 \Rightarrow f\cdot e = 0)$.

(iii) $\forall e \in E$, $\forall x \in S$, $(e\cdot x = 0 \Rightarrow x\cdot e = 0)$.

(iv) $\forall r \in \mathbb{N}$, $\forall x \in S$, $(x^r = 0 \Rightarrow x = 0)$.

(v) $\forall x \in S$, $(x^2 = 0 \Rightarrow x = 0)$.

(vi) $\forall x,y \in S$, $(x \cdot y = 0 \Rightarrow y\cdot x = 0)$.

(vii) $(S,\cdot)$ is inverse.

Example 7: (i) Take $(R,+,\cdot)$ to be a regular ring in which $(R,\cdot)$ is not orthodox. Set $S = R \cup \{a\}$ where $a \notin R$ and define $r + a = a + r = r$, $a + a = a = r\cdot a = a\cdot r$ for all $r \in R$. Then $(S,+,\cdot)$ is a semiring in which $(S,\cdot)$ is regular and a is the additive and multiplicative zero of S. Hence $(S,+,\cdot)$ satisfies condition (v) of Result 6, but is not orthodox.

(ii) Let $(S,+)$ be a semilattice with $|S| \geq 2$ and define $x\cdot y = x$ for all $x,y \in S$. Then $(S,+,\cdot)$ is an additively inverse semiring in which the multiplicative semigroup is orthodox but not inverse.

Lallement ([8] Theorem 4.6) has proved that a primitive regular ring-semigroup is a group with zero adjoined. In particular, a completely-0-simple ring-semigroup is a group with zero adjoined.

Define a partial order on the set of idempotents E of a semigroup S by: $f \leq e$ if and only if $f = e \cdot f \cdot e$. A nonzero idempotent e is primitive in S if for $f \in E$, $0 \neq f \leq e$ implies $f = e$. The semigroup S satisfies Min - E if the minimum condition holds for E under the specified order; Max - E is defined dually. If $x \in S$ let

$$J(x) = \{x\} \cup xS \cup Sx \cup SxS$$

denote the principal (two-sided) ideal generated by x, and

$$I(x) = \{y \in J(x) : J(y) \subsetneq J(x)\}$$

the set of nongenerators of J(x). Then S is called completely semisimple if for each nonzero $x \in S$, the Rees quotient semigroup $J(x)/I(x)$ contains a primitive idempotent, in which case every nonzero idempotent of $J(x)/I(x)$ is primitive. We let Min - J signify the minimum condition on the set of principal ideals of S; Max - J is its dual.

A ring is semiprime if it contains no nonzero nilpotent (one-sided) ideals, and artinian if it satisfies the minimum condition on right ideals. A ring is atomic if it is a (direct) sum of minimal right ideals.

As a generalization of Lallement's theorem we have the following result.

Result 8: ([5] Theorem 4.)

For a semigroup S, the following conditions are equivalent:

(i) S is completely semisimple and satisfies Min - J.

(ii) S is completely semisimple and satisfies Min - E.

(iii) S is regular and satisfies Min - E.

Furthermore, if S is a ring-semigroup, then (i), (ii) and (iii) are equivalent to each of the following conditions:

(iv) (S,+,·) is a semiprime atomic ring.

(v) (S,+,·) is a direct sum of dense rings of finite-rank linear transformations of vector spaces over division rings.

Example 9: Whilst the equivalent conditions (i),(ii),(iii) of Result 8 imply that S is regular with Min - J, the converse does not hold, even for rings with identity. To see this, consider the full ring of linear transformations of an infinite-dimensional vector space. This ring is regular ([9], Theorem 7.3) with Min - J ([10], Theorem 1.4.2) but does not satisfy Min - E, since the projections onto an infinite descending chain of subspaces give rise to an infinite descending chain of idempotents.

Result 10: ([5] Theorem 5.)

For a semigroup S, each of the following conditions implies the next.

(i) S is completely semisimple and satisfies Max - J.

(ii) S is completely semisimple and satisfies Max - E.

(iii) S is regular and satisfies Max - E.

Furthermore, if S is a ring-semigroup, then conditions (i), (ii) and (iii) are equivalent to each other and to the condition:

(iv) (S,+,·) is a semiprime artinian ring i.e. a finite direct sum of full matrix rings over division rings.

Example 11: (i) The bicyclic semigroup $\beta(p,q) = \langle p,q : pq = 1 \neq qp \rangle$ is regular and satisfies Max - E but not Min - E ([1] Theorem 2.53). Moreover it is not completely semisimple. Thus in Theorem 10, condition (iii) does not imply condition (ii) for non-ring-semigroups.

(ii) Let C_n be the chain of length n, $n \geq 2$. Suppose these chains have a common zero element 0. Take E to be the 0-direct union of C_n, $n \geq 2$. Then E is a semilattice satisfying Max - E and

Min - E. The Munn semigroup, T_E of E ([6] Section V.4) is an inverse semigroup with E as its semilattice of idempotents (and thus is completely semisimple by Theorem 8) but does not satisfy Max - J.

Thus in Theorem 10, (ii) $\Longrightarrow$ (i) is not valid for non-ring-semigroups.

(iii) [2] Examples (a),(b) page 805 give examples of regular ring-semigroups which:

(a) have only two principal ideals but do not satisfy Max - E or Min - E,

(b) are completely semisimple but do not satisfy Max - J, Min - J, Max - E or Min - E.

3. <u>Congruences on regular semirings</u>: Semirings in which the additive semigroup is inverse and the multiplicative semigroup is regular (and hence the additive semigroup is a semilattice of abelian groups) are considered in [11],[13]. These papers also consider the case in which the multiplicative semigroup is simple or 0-simple.

<u>Result 12</u>: ([4])

<u>In a semiring</u> (S,+,·) <u>the additive Green s relations</u> $\mathcal{L},\mathcal{R},\mathcal{H},\mathcal{D},\mathcal{J}$ <u>are congruences on the multiplicative semigroup</u> (S,·).

A semigroup (S,·) is said to be <u>congruence-free</u> if the only congruence relations on S are 1_S and $S\times S$. Thus a congruence-free semigroup is simple or 0-simple, since if I is an ideal of S, ρ_I defined by $\rho_I = (I\times I)\cup 1_S$, is a congruence relation on S.

A band (S,·) is <u>left</u> (<u>right</u>) regular if axa = ax (axa = = xa) for all $a,x\in S$.

Lemma 13: Take $(S,\cdot)$ to be a regular semigroup on which $\mathcal{L}(\mathcal{R})$ is trivial. Then S is a right (left) regular band i.e. a semilattice of left [right] zero semigroups.

Proof: Take $x,a \in S$ and $a' \in V(a)$. Then $a \mathcal{L} a'a$ and thus $a = a'a$. Hence $a^2 = a(a'a) = a$. Now $S^1axa \subseteq S^1xa = S^1xaxa \subseteq$ $\subseteq S^1axa$ and so $S^1axa = S^1xa$ i.e. $axa \mathcal{L} xa$. Thus $axa = xa$ for all $a,x \in S$. □

Corollary 14: Take $(S,\cdot)$ to be a regular semigroup on which $\mathcal{D}$ is trivial. Then $(S,\cdot)$ is a semilattice.

Proof: Since $\mathcal{L} = \mathcal{R} = 1_S$, S is both a left and right regular band and hence a semilattice. □

A semiring $(S,+,\cdot)$ is said to be completely simple if the additive semigroup is completely simple and the multiplicative semigroup ie either completely simple or completely 0-simple.

Theorem 15: ([13] Theorem 24). Take $(S,+,\cdot)$ to be a completely simple semiring.

(i) If the multiplicative semigroup is completely 0-simple, then the semiring is a division ring.

(ii) If the multiplicative semigroup is completely simple, then the additive semigroup is a rectangular band and the multiplicative semigroup is a product of two completely simple semigroups $S = I \times \Lambda$ and the operations on the semiring S are given by

$$(i,\lambda) + (j,\mu) = (i,\mu)$$
$$(i,\lambda)\cdot(j,\mu) = (i\cdot j, \lambda\cdot\mu)$$

for all $i,j \in I$, λ, $\mu \in \Lambda$.

Theorem 16: Take $(S,+,\cdot)$ to be a semiring in which the additive semigroup is regular and the multiplicative semigroup is

congruence-free. Then the additive semigroup is either a group, a semilattice, a left zero band or a right zero band.

Proof: By [3] Lema 2 (i), the set $E^{[+]}$ is an ideal of $(S,\cdot)$. Since $(S,\cdot)$ is simple or 0-simple, $E^{[+]} = \{0\}$ or $E^{[+]} = S$. Since $(S,+)$ is a regular semigroup, it is either a group or a band.

Because $\mathcal{T}$ is a congruence on the multiplicative semigroup $(S,\cdot)$, $\mathcal{T} = 1_S$ or $\mathcal{T} = S \times S$.

(i) In the case $\mathcal{T} = 1_S$, then $(S,+)$ is a semilattice by Corollary 14, since $\mathcal{D} \subseteq \mathcal{T}$.

(ii) When $\mathcal{T} = S \times S$, $(S,+)$ is a simple semigroup.

(a) $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = 1_S$.

Then $(S,+)$ is a simple semilattice and thus the trivial group.

(b) $\mathcal{H} = \mathcal{L} = 1_S$, $\mathcal{R} = \mathcal{D} = S \times S$.

Then $(S,+)$ is right simple and a band. Thus, by [1] Theorem 1.27, S is the direct product of a group and a right zero band and thus is a right zero band since $\mathcal{H} = 1_S$.

(c) $\mathcal{H} = \mathcal{R} = 1_S$, $\mathcal{L} = \mathcal{D} = S \times S$.

By symmetry, $(S,+)$ is a left zero semigroup.

(d) $\mathcal{H} = S \times S$.

In this case $(S,+)$ is a group. □

Example 17: We provide examples of semirings in which the additive semigroup is regular and the multiplicative semigroup is congruence-free, as in Theorem 16.

(i) Take $(S,+,\cdot)$ to be the two element field. Then $(S,\cdot)$ is congruence-free. Here $(S,+)$ is a group.

(ii) The two-element chain has as its multiplicative semigroup a congruence-free semigroup. Here $(S,+)$ is a semilattice.

(iii) Take $(S,\cdot)$ to be any congruence-free semigroup. Define the binary operation $+: S\times S \to S$ by $x + y = x$ for all $x,y \in S$. Then $(S,+,\cdot)$ is a semiring in which the additive semigroup is a left-zero band.

Theorem 18: Take $(S,+,\cdot)$ to be a semiring in which $(S,+)$ is a regular semigroup and $(S,\cdot)$ has a unique non-trivial congruence. Then the additive semigroup is either a group, a semilattice of groups, a semilattice of left zero bands, a semilattice of right zero bands, a left group or a right group.

Proof: Denote by ρ the non-trivial congruence on $(S,\cdot)$. Since $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{T}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{T}$, we have that either $\mathcal{L} \subseteq \mathcal{R}$ or $\mathcal{R} \subseteq \mathcal{L}$. We shall only consider the cases in which $\mathcal{L} \subseteq \mathcal{R}$, since the results for $\mathcal{R} \subseteq \mathcal{L}$ will follow by symmetry.

(i) $\mathcal{L} = \mathcal{R} = \mathcal{D} = 1_S$.

By Corollary 14, $(S,+)$ is a semilattice.

(ii) $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{T}$.

Clearly, a regular semigroup in which $\mathcal{L} = \mathcal{R}$ is a semilattice of groups.

(iii) $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \rho \subset \mathcal{T} = S\times S$.

In this case, $(S,+)$ is a semilattice of groups and also simple since $\mathcal{T} = S\times S$. Hence $(S,+)$ is a group.

We now consider the case in which $\mathcal{L} \subset \mathcal{R}$.

(iv) $1_S = \mathcal{H} = \mathcal{L} \subseteq \mathcal{R} = \mathcal{D} \subseteq \mathcal{T} \subseteq S\times S$.

Since $\mathcal{L}$ is trivial, by Lemma 13, S is a right regular band, i.e. a semilattice of right zero semigroups.

The other cases were considered in Theorem 16 or follow by symmetry.

References

[1] A.H. CLIFFORD and G.B. PRESTON (1961), The algebraic theory of semigroups, Vol. 1 (American Mathematical Society. Mathematical Surveys 7).

[2] K.R. GOODEARL and D. HANDELMAN (1975), Simple self-injective rings, Comm. Algebra 3(9), 797-834.

[3] M.O.P. GRILLET (1970), Subdivision rings of a semiring, Fund. Math, 67, 67-74.

[4] M.O.P. GRILLET (1975), Semirings with a completely simple additive semigroup, J. Austral. Math. Soc. 20, 257-267.

[5] J. HANNAH, J.S. RICHARDSON and J. ZELEZNIKOW (1980), Completely semisimple ring-semigroups, J. Austral. Math. Soc. 30, 150-156.

[6] J.M. HOWIE (1976), An introduction to semigroup theory, Academic Press, Inc., London.

[7] P.H. KARVELLAS (1974), Inversive semirings, J. Austral. Math. Soc. 18, 277-287.

[8] G. LALLEMENT (1967), Demi-groups reguliers, Annali mat. pura. appl. 77, 47-130.

[9] N.H. McCOY (1964), The theory of rings (Macmillan, New York).

[10] M. PETRICH (1974), Rings and semigroups (Lecture Notes in Mathematics 380, Springer-Verlag, Berlin,Heidelberg, New York).

[11] H.J. WEINERT (1983), On 0-simple semirings, semigroup semirings and two kinds of division semirings (to appear).

[12] J. ZELEZNIKOW (1980), Orthodox semirings and rings, J. Austral. Math. Soc. 30, 50-54.

[13] J. ZELEZNIKOW (1981), Regular semirings, Semigroup forum 23, 119-136.

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
United States of America

(Oblatum 17.11. 1983)