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COMPLETELY REGULAR MODIFICATION AND PRODUCTS

Petr SIMON

Abstract: If X is a topological space, denote CR(X) the completely regular modification of X. The aim of the present paper is to give an example of two T_3-spaces X, Y such that $CR(X\times Y) \neq CR(X) \times CR(Y)$.

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There is a plenty of papers dealing with the commutativity of products and a suitable functor from the category of topological spaces into itself. To the author's knowledge, the functor of completely regular modification has been investigated from this point of view in [O] and [P]. For a topological space X, denote CR(X), the completely regular modification of X, the space whose underlying set is the same as that of X, equipped with the topology, the base of which consists of all cozero subsets of X. It is easy to show that CR(X) has the largest completely regular topology contained in the topology of X. Let us remind the best results concerning the commutativity of CR and products:

Theorem [O]: Let X be Tychonoff. Then the following are equivalent:

(i) X is locally compact,

(ii) for each space Y, $CR(X \times Y) = X \times CR(Y)$.

<u>Theorem</u> [0]: Let X be a topological space and suppose that $CR(X)$ is not locally compact. Then there exists a Hausdorff space Y such that $CR(X \times Y) \neq CR(X) \times CR(Y)$.

According to these two theorems, the picture is pretty clear: local compactness is the crucial property. Unfortunately, the proof of the second theorem mentioned above essentially uses the fact that the space Y is not regular.

We do not know the answer, whether "Hausdorff" can be replaced by "regular" in the second theorem of S. Oka. Nevertheless, we can exhibit the following

<u>Example</u>: There exist regular spaces X and Y such that $CR(X) \times CR(Y) \neq CR(X \times Y)$.

The idea is fairly simple. Let us start with a completely regular, non-normal space T, let $A, B \subseteq T$ be the two closed disjoint sets which cannot be separated. Run the space T through the Jones machine. You will obtain the regular space X which contains a point p and a closed set A_0 isomorphic to A such that p and A_0 cannot be functionally separated. This implies that whenever U is a cozero set in X which contains p, then $U \cap A_0$ is infinite. Consequently, the point (p,p) belongs to the closure of the set $\{(x,x): x \in A_0\}$ in the space $CR(X) \times CR(X)$. In order to show that $CR(X \times X)$ differs from $CR(X) \times CR(X)$, we need to find a continuous real-valued function on $X \times X$ which vanishes in (p,p) and equals 1 in each (x,x), $x \in A_0$.

Unfortunately, this does not work in general and we ought to be a bit more careful when choosing the starting non-normal space - in fact, we shall need two such spaces. In spite of this, the idea has just been fully described and the rest are mere technical complications.

A. <u>The modified Tychonoff plank</u>. Let $\tau \le 2^{\omega}$ be a cardinal number, let $\mathcal{F} = \{F_{\alpha} : \alpha \in \tau\}$ be an arbitrary family of infinite subsets of ω.

The modified Tychonoff plank $T(\mathcal{F})$ is defined as follows: The underlying set is $(\tau + 1) \times (\omega + 1) - \{(\tau, \omega)\}$, every point (α, n) (for $\alpha < \tau$, $n < \omega$) is isolated, the neighborhood base of a point (τ, n) (for $n < \omega$) is the collection $\{\{(\tau, n)\} \cup \{(\alpha, n) : \alpha \in \tau - C\} : C \in [\tau]^{\le \omega}\}$, the neighborhood base of a point (α, ω) (for $\alpha < \tau$) is the collection $\{\{(\alpha, \omega)\} \cup \{(\alpha, n) : n \in F_{\alpha} - F\} : F \in [\omega]^{<\omega}\}$. Sometimes it will be convenient to emphasize by a subscript $(\alpha, n)_{\mathcal{F}}$ that the pair (α, n) belongs to $T(\mathcal{F})$.

Now, the space $T(\mathcal{F})$ is completely regular Hausdorff 0-dimensional. It is normal if and only if $|\mathcal{F}| \le \omega$, because the sets $A_{\mathcal{F}} = \{\tau\} \times \omega$ and $B_{\mathcal{F}} = \tau \times \{\omega\}$ cannot be separated iff $\tau > \omega$.

The forthcoming lemma shows one important property of continuous functions on $T(\mathcal{F})$.

For $\mathcal{F} \subseteq [\omega]^{\omega}$, denote $\mathcal{J}(\mathcal{F}) = \{X \in [\omega]^{\omega} : |\{F \in \mathcal{F} : |F \cap X| = \omega\}| \le \omega\}$.

<u>Lemma 1</u>. Let $\mathcal{F} \subseteq [\omega]^{\omega}$, $\tau = |\mathcal{F}| > \omega$, let $f : T(\mathcal{F}) \to \mathbb{R}$ be continuous, $\varepsilon > 0$. Then

(i) if $|\{\alpha \in \tau : |f((\alpha, \omega))| \ge \varepsilon\}| \le \omega$, then $\{n \in \omega : |f((\tau, n))| > \varepsilon\} \in \mathcal{J}(\mathcal{F})$, and almost conversely

(ii) if $\{n \in \omega : |f((\tau, n))| \ge \varepsilon\} \in \mathcal{J}(\mathcal{F})$, then $|\{\alpha \in \tau : |f((\alpha, \omega))| > \varepsilon\}| \le \omega$.

Proof. Since f is continuous, then for each $n, k \in \omega$ the set $S_{n,k} = \{\alpha \in \tau : |f((\alpha,n)) - f((\tau,n))| \geq \frac{1}{k}\}$ is countable. Let $S = \bigcup_{k=1}^{\infty} \bigcup_{m=0}^{\infty} S_{n,k}$, $Z = \tau - S$. Then for $\alpha \in Z$ and $n \in \omega$ $f((\alpha,n)) = f((\tau,n))$.

(i) Denote $M = \{n \in \omega : |f((\tau,n))| > \varepsilon\}$. If $\alpha \in Z$ is such that $|M \cap F_\alpha| = \omega$, then the continuity of f implies $|f((\alpha,\omega))| \geq \inf\{|f((\alpha,n))| : n \in F_\alpha \cap M\} = \inf\{|f((\tau,n))| : n \in F_\alpha \cap M\} \geq \varepsilon$. Therefore $\{\alpha \in \tau : |F_\alpha \cap M| = \omega\} \subseteq \{\alpha \in \tau : |f((\alpha,\omega))| \geq \varepsilon\} \cup S$. Since both sets on the right-hand side are at most countable, $M \in \mathcal{J}(\mathcal{F})$, which was to be proved.

(ii) Denote $N = \{n \in \omega : |f((\tau,n))| \geq \varepsilon\}$. If $\alpha \in Z$ is such that $|F_\alpha \cap N| < \omega$, then $|f((\alpha,\omega))| \leq \sup\{|f((\alpha,n))| : n \in F_\alpha - N\} = \sup\{|f((\tau,n))| : n \in F_\alpha - N\} \leq \varepsilon$. Thus $\{\alpha \in \tau : |f((\alpha,\omega))| > \varepsilon\} \subseteq \{\alpha \in \tau : |F_\alpha \cap N| = \omega\} \cup S$. Since $N \in \mathcal{J}(\mathcal{F})$, the set $\{\alpha \in \tau : |F_\alpha \cap N| = \omega\}$ is at most countable, hence the set $\{\alpha \in \tau : |f((\alpha,\omega))| > \varepsilon\}$ is at most countable, too. □

B. <u>Jones machine</u>. A well-known construction, the final form of which is due to F.B. Jones, goes as follows [J]: Let T be a non-normal space, denote $A, B \subseteq T$ the closed, disjoint sets which cannot be separated. Let $Z = (T \times \omega) \cup \{p\}$, where $p \notin T \times \omega$. The topology on Z is the usual product topology in all points other than p, the basic neighborhood of p is $\{p\} \cup (T \times (\omega - k))$, where $k \in \omega$. Define an equivalence relation $\sim$ on Z by $(x,n) \sim (y,m)$ iff either $x \in A$, $y = x$ and $n = 2k+1$, $m = 2k+2$, or $x \in B$, $y = x$ and $n = 2k$, $m = 2k+1$. The space $J(T)$ is the quotient space Z modulo $\sim$.

The basic properties of $J(T)$ are the following: If T is regular (resp. Hausdorff, resp. T_1), then $J(T)$ is, but $J(T)$ is not completely regular, because the point p cannot be functionally separated from the closed set $A \times \{0\}$.

For the modified non-normal Tychonoff plank $T(\mathcal{F})$, denote $A = \{(\tau, n) : n \in \omega\}$, $B = \{(\alpha, \omega) : \alpha \in \tau\}$ and consider the space $J(T(\mathcal{F})) = J(\mathcal{F})$. (If necessary, we shall again denote the points of $J(\mathcal{F})$ as $p_{\mathcal{F}}$ and $((\alpha, n), k)_{\mathcal{F}}$.) Then the following holds.

<u>Lemma 2</u>. Let $\mathcal{F} \subseteq [\omega]^{\omega}$ be uncountable, let $f : J(\mathcal{F}) \to \mathbb{R}$ be continuous, $f(p_{\mathcal{F}}) = 0$, $\varepsilon > 0$. Then

$\{n \in \omega : |f(((\tau, n), 0))| > \varepsilon\} \in \mathcal{J}(\mathcal{F})$.

<u>Proof</u>. There is some $k \in \omega$ such that for all $x \in \{p\} \cup \cup (T(\mathcal{F}) \times (\omega - k))/\sim$, $|f(x)| < \varepsilon/2$. Hence there is some even $j \geq k$ such that $|f(x)| < \varepsilon/2$ for all $x \in B \times \{j\}$.

Choose $\delta > 0$, $\delta < \varepsilon/2.j$. Since for each $x \in B \times \{j\}$, $|f(x)| < \varepsilon/2$, by Lemma 1,(i), the set $\{n \in \omega : |f(((\tau, n), j))| > \varepsilon/2\} \in \mathcal{J}(\mathcal{F})$. Since $A \times \{j\}$ was identified with $A \times \{j - 1\}$, the set $\{n \in \omega : |f(((\tau, n), j-1))| > \varepsilon/2\}$ belongs to $\mathcal{J}(\mathcal{F})$, too. Thus $\{n \in \omega : |f(((\tau, n), j-1))| \geq \varepsilon/2 + \delta\} \in \mathcal{J}(\mathcal{F})$, by Lemma 1, (II), the set $\{\alpha \in \tau : |f(((\alpha, \omega), j-1))| > \varepsilon/2 + \delta\}$ is at most countable. By the identification, $\{\alpha \in \tau : |f(((\alpha, \omega), j-2))| > \varepsilon/2 + \delta\}$ is countable, too, and the same holds for $\{\alpha \in \tau : |f(((\alpha, \omega), j-2))| \geq \varepsilon/2 + 2\delta\}$. Proceeding further, we obtain finally that $\{n \in \omega : |f(((\tau, n), 0))| > \varepsilon/2 + j.\delta\} \in \mathcal{J}(\mathcal{F})$, which was to be proved, as $\varepsilon/2 + j.\delta < \varepsilon$. $\square$

C. <u>How to do it</u>. The forthcoming lemma is fully proved in [S].

<u>Lemma 3</u>. There is an infinite maximal almost disjoint family $\mathcal{M} \subseteq [\omega]^{\omega}$ which admits a disjoint partition $\mathcal{M} = \mathcal{F} \cup \mathcal{G}$ such that $\mathcal{J}(\mathcal{M}) = \mathcal{J}(\mathcal{F}) = \mathcal{J}(\mathcal{G})$.

Notice that both the collections $\mathcal{F}$, $\mathcal{G}$ must be uncountable. Suppose the contrary, let $\mathcal{F} = \{F_n : n \in \omega\}$. Choose a countably infinite subset $\mathcal{G}' \subseteq \mathcal{G}$ and enumerate it as $\{G_n : n \in \omega\}$ in such a way that for each $G \in \mathcal{G}'$, the set $\{n \in \omega : G = G_n\}$ is infinite. Then pick up inductively $k_n \in G_n - \bigcup_{i=0}^{n} F_i$, $k_n > k_{n-1}$. Now the set $K = \{k_n : n \in \omega\}$ belongs to $\mathcal{J}(\mathcal{F})$, for $K \cap F$ is finite for each $F \in \mathcal{F}$. On the other hand, the set $\{M \cap K : M \in \mathcal{M}$ and $|M \cap K| = \omega\}$ is an infinite maximal almost disjoint family on K, hence it cannot be countable. Thus $K \in \mathcal{J}(\mathcal{F})$, $K \notin \mathcal{J}(\mathcal{M})$, which contradicts the lemma.

The spaces we promised to construct, are $X = J(\mathcal{F})$, $Y = J(\mathcal{G})$, where $\mathcal{F}$ and $\mathcal{G}$ are as in Lemma 3. Let $\tau = |\mathcal{F}|$, $\mu = |\mathcal{G}|$; using the notation as before, denote

$$\Delta = \{(((\tau,n),0)_{\mathcal{F}}, ((\mu,n),0)_{\mathcal{G}}) : n \in \omega\}.$$

First, we shall prove that the point $(p_{\mathcal{F}}, p_{\mathcal{G}})$ is a cluster point of Δ in $CR(X) \times CR(Y)$.

Indeed, choose arbitrarily a cozero set U with $p_{\mathcal{F}} \in U \subseteq J(\mathcal{F})$, and a cozero set V with $p_{\mathcal{G}} \in V \subseteq J(\mathcal{G})$. By Lemma 2, $K = \{n \in \omega : (\tau,n)_{\mathcal{F}} \notin U\} \in \mathcal{J}(\mathcal{F})$ and $L = \{n \in \omega : (\mu,n)_{\mathcal{G}} \notin V\} \in \mathcal{J}(\mathcal{G})$. By Lemma 3, $\mathcal{J}(\mathcal{F}) = \mathcal{J}(\mathcal{G}) = \mathcal{J}(\mathcal{M})$, and clearly $\mathcal{J}(\mathcal{M})$ is a proper ideal on ω, thus $\omega - K \cup L$ is infinite. Clearly, for $n \in \omega - K \cup L$, $(((\tau,n),0)_{\mathcal{F}}, ((\mu,n),0)_{\mathcal{G}}) \in U \times V$. Thus each neighborhood of a point $(p_{\mathcal{F}}, p_{\mathcal{G}})$ in $CR(X) \times CR(Y)$ meets Δ, which was to be proved.

Second, we shall separate the point $(p_{\mathcal{F}}, p_{\mathcal{G}})$ from Δ in the space $CR(X \times Y)$.

Define a function $f : X \times Y \longrightarrow \mathbb{R}$ as follows: $f((x,y)) = 1$ provided that there are $n \in \omega$, $\alpha \in \tau + 1$ and $\beta \in \mu + 1$ such that $x = ((\alpha,n),0)_{\mathcal{F}}$, $y = ((\beta,n),0)_{\mathcal{G}}$, otherwise $f((x,y)) = 0$.

Clearly, $f \upharpoonright \Delta \equiv 1$, $f((p_{\mathcal{F}}, p_{\mathcal{G}})) = 0$, thus it remains to check that f is continuous.

Pick up $(x,y) \in X \times Y$. Then there are only four non-trivial cases:

1. $x = ((\alpha,\omega),0)_{\mathcal{F}}$ for $\alpha < \tau$,

   $y = ((\beta,\omega),0)_{\mathcal{G}}$ for $\beta < \mu$.

Let $U = \{x\} \cup \{((\alpha,n),i)_{\mathcal{F}} : n \in F_\alpha - G_\beta,\ i \in \{0,1\}\}$,
$V = \{y\} \cup \{((\beta,n),i)_{\mathcal{G}} : n \in G_\beta - F_\alpha,\ i \in \{0,1\}\}$.

Since $\mathcal{M}$ was assumed to be almost disjoint, $(F_\alpha - G_\beta) \cap \cap (G_\beta - F_\alpha) = \emptyset$, thus $f \upharpoonright U \times V \equiv 0$.

2. $x = ((\alpha,\omega),0)$ for $\alpha < \tau$,

   $y = ((\beta,n),0)$ for $\beta \leq \mu$, $n < \omega$.

Let $U = \{x\} \cup \{((\alpha,m),i) : m \in F_\alpha - \{n\},\ i \in \{0,1\}\}$,
$V = \{y\} \cup \{((\gamma,n),0) : \gamma < \mu\}$.

Then $f \upharpoonright U \times V \equiv 0$.

3. $x = ((\alpha,n),0)$ for $\alpha \quad \tau$, $n < \omega$

   $y = ((\beta,\omega),0)$ for $\beta < \mu$.

This case is symmetrical to the previous one.

4. $x = ((\alpha,n),0)$ for $\alpha \leq \tau$, $n < \omega$,

   $y = ((\beta,m),0)$ for $\beta \leq \mu$, $m < \omega$.

Let $U = \{x\} \cup \{((\delta,n),0) : \delta < \tau\}$,
$V = \{y\} \cup \{((\gamma,m),0) : \gamma < \mu\}$.

Then if $f(x,y) = 0$, which takes place if $n \neq m$, we have $f \upharpoonright U \times V \equiv 0$, and if $n = m$, then $f \upharpoonright U \times V \equiv 1$.

In any case other than these just mentioned, the existence of neighborhoods U, V with $f \upharpoonright U \times V \equiv 0$, is obvious.

Thus f is a continuous function which separates $(p_{\mathcal{F}}, p_{\mathcal{G}})$ and Δ.

<u>Remark.</u> The spaces we have constructed, are regular. One

can want, moreover, that both X, Y have a base consisting of interiors of zero sets. It suffices to start with $T(\mathcal{F})$ and $T(\mathcal{G})$ as before, but then adopt the construction described in [W] instead of Jones machine.

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Matematický ústav Univerzity Karlovy, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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