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POINTLESS UNIFORMITIES II. (DIA)METRIZATION

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Abstract: Metrization theorems for pointless uniformities and weak uniformities are proved.

Key words: Uniformity on a locale, diameters, metrization.

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This paper is a lose continuation of the paper [6]. There we have proved the equivalence of complete regularity and uniformizability in locales and indicated a role of diameters. A system $\mathfrak{D}$ of diameters gives rise to a uniformity (or, to a weak uniformity, according to how strong conditions are imposed on the diameters) $\mathcal{U}(\mathfrak{D})$. We have seen, in particular, that if a locale is uniformizable at all, it is uniformizable by a $\mathcal{U}(\mathfrak{D})$. The main aim of this paper is to prove metrization theorems for pointless uniformities, i.e. to show that, in fact, each uniformity on a locale is a $\mathcal{U}(\mathfrak{D})$, and that it is induced by a single diameter function whenever it has a countable basis. This goal is achieved by modifying the standard metrization argument (see, e.g., [5]) and, perhaps, yields also a better insight into what is going on there.

The first, and larger, part of the article (Sections 1-3) is devoted to a discussion of various conditions one can impose on diameter functions. Section 1 contains the basic definitions and relations between the conditions. In Section 2 it is shown that in the spatial case, the metric diameters are in a natural one-one correspondence with the pseudometrics. Section 3 deals with constructions allowing to obtain stronger properties of diameters. In the last, fourth, section the induced uniformities are discussed and metrization theorems are proved.

The terminology follows the standard usage (as, e.g., in [4], [1]), in special definitions the notation and convention of [6] are preserved (with the exception of the condition (M) which now contains automatically the condition (A)).

1. Diameters

1.1. We say that a subset of a locale L is <u>connected</u> if $\forall a,b \in S \quad \exists a_1,\dots,a_n \in S$ such that $a_1 = a$, $a_n = b$ and

$a_i \wedge a_{i+1} \neq 0$ for $i = 1,\dots,n-1$.

We say that it is <u>strongly connected</u> if

$a,b \in S \Rightarrow a \wedge b \neq 0$.

The system of all connected subsets of L will be denoted by

$$\mathrm{conn}(L),$$

that of the strongly connected ones by

$$\mathcal{S}(L).$$

1.2. A <u>pre-diameter</u> on a locale L is a function

$$d: L \longrightarrow \mathbb{R}_+$$

($\mathbb{R}_+$ is the set of the non-negative reals) such that

(i) $d(0) = 0$,

(ii) $a \leq b \Rightarrow d(a) \leq d(b)$,

(iii) $\forall \varepsilon > 0$, $\{a \mid d(a) < \varepsilon\}$ is a cover of L.

It is said to be <u>continuous</u> if, moreover,

(C) for each monoton- (= linearly ordered by $\leq$) $S \subset L$,

$$d(\bigvee S) = \sup\{d(a) \mid a \in S\}.$$

1.3. A pre-diameter d is said to be

- a <u>weak diameter</u> if

(W): for a, b such that $a \wedge b \neq 0$,

$$d(a \vee b) \leq 2 \max(d(a), d(b));$$

- an <u>additive diameter</u> if

(A): for a,b such that $a \wedge b \neq 0$,

$$d(a \vee b) \leq d(a) + d(b);$$

- a <u>star diameter</u> if

($\ast$): for $S \in \mathcal{S}(L)$

$$d(\bigvee S) \leq 2 \sup\{d(a) \mid a \in S\};$$

- a <u>star-additive diameter</u> if

($\ast$A): for $S \in \mathcal{S}(L)$,

$$d(\bigvee S) \leq \sup\{d(a) + d(b) \mid a, b \in S,\ a \neq b\};$$

- a <u>strong diameter</u> if

(S): for $S \in \operatorname{conn}(L)$,

$$d(\bigvee S) \leq \sup\{\inf\{\sum_{i=1}^{n} d(a_i) \mid a_i \in S,\ a_1 = a,\ a_n = b,\ a_i \wedge a_{i+1} \neq 0\} \mid a, b \in S,\ a \neq b\};$$

- a <u>metric diameter</u> if

(M): (A) and

$$\forall x \in L \quad \forall \varepsilon > 0 \quad \exists\, u, v,\ u \wedge x \neq 0 \neq v \wedge x \ \&\ d(u), d(v) < \varepsilon \ \&\ d(u \vee v) > d(x) - \varepsilon .$$

1.4. <u>Remark</u>: The following implications are obvious

$$\begin{array}{ccccc} S & \Longrightarrow & (\ast A) & \Longrightarrow & (\ast) \\ & & \Downarrow & & \Downarrow \\ & & (A) & \Longrightarrow & (W). \end{array}$$

In [6, Lemma 5.1] we have seen that (M) $\Longrightarrow$ ($\ast$). In fact, as we will shortly see, (M) is the strongest of all the mentioned requirements (and, moreover, implies continuity). In the next section we will show that the metric diameters correspond in the spatial case exactly to the pseudometrics. Thus, they can be understood as a natural modification of the notion of distance for the purposes of general locales.

The reason why we list the other mentioned conditions on pre-diameters (and no further ones used elsewhere, e.g. in [2]) is, of course, given by the aims of the article. The condition (W) is the weakest one one needs to induce at least weak uniformities; (A) is very natural and, besides, it is a part of (M); ($*$A) is also very natural, probably the most intuitive of all, and it will play a technical role: a star-additive diameter can be very satisfactorily approximated by a metric one: (S) is an extension of ($*$A) and will appear as a consequence of (M). The condition ($*$) is about the minimum one needs for generating uniformities; besides, star-diameters will also play a technical role.

1.5. <u>Theorem:</u> A metric diameter is a continuous strong diameter.

<u>Proof:</u> (C): Let $S \subset L$ be monotone. Take an $\varepsilon > 0$ and choose u, v such that $d(u), d(v) < \varepsilon$, $u \wedge \bigvee S \neq 0 \neq v \wedge \bigvee S$ and $d(u \vee v) > d(\bigvee S) - \varepsilon$. We have $x, y \in S$ such that $x \wedge u \neq 0 \neq y \wedge v$. If, say, $y \geq x$, we have also $u \wedge y \neq 0$. Thus,

$$d(u \vee v) \leq d(y \vee u \vee v) \leq d(y \vee u) + \varepsilon \leq d(y) + 2\varepsilon$$

so that

$$d(y) > d(\bigvee S) - 3\varepsilon .$$

Hence, $\sup \{d(y) \mid y \in S\} \geq d(\bigvee S)$. On the other hand, obviously $\sup d(y) \leq d(\bigvee S)$.

(S): Let this not hold. Then, we have an $S \in$ conn (L) and an $\eta > 0$ such th-

$$d(\bigvee S) > \sup\{\inf \sum_{i=1}^{n} d(a_i) \mid a_i \in S,\ a_1 = a,\ a_n = b,$$
$$a_i \wedge a_{i+1} \neq 0\} \mid a, b \in S,\ a \neq b\} + \eta\ .$$

Take an $\varepsilon > 0$ such that $\varepsilon < \frac{1}{6}\eta$ and choose u, v such that

$d(u), d(v) < \varepsilon$, $u \wedge \bigvee S \neq 0 \neq v \wedge \bigvee S$ and $d(u \vee v) > d(\bigvee S) - \varepsilon$.

Consider $a, b \in S$ with $u \wedge a \neq 0 \neq v \wedge b$.

I. Let $a \neq b$. Then we have, in particular,

$$d(\bigvee S) > \inf\{\sum_{i=1}^{n} d(a_i) \mid a_i \in S,\ a_1 = a,\ a_n = b,$$
$$a_i \wedge a_{i+1} \neq 0\} + \eta$$

and hence there are $a_1 = a, a_2, \dots, a_n = b$, $a_i \wedge a_{i+1} \neq 0$ such that

(1) $\quad d(\bigvee S) > \sum d(a_i) + \frac{1}{2}\eta\ .$

By 1.2(iii) we can choose $u_i \in L$ such that

$$d(u_i) < \varepsilon \quad \text{and } u_i \leq a_i \wedge a_{i+1}.$$

We obtain

$$d(u \vee u_1) \leq d(u \vee a_1) \leq d(a_1) + \varepsilon\ ,$$
$$d(u_1 \vee u_2) \leq d(a_2),$$
$$\vdots$$
$$d(u_{n-2} \vee u_{n-1}) \leq d(a_{n-1}),$$
$$d(u_{n-1} \vee v) \leq d(a_n \vee v) \leq d(a_n) + \varepsilon\ .$$

Using repeatedly (A) we obtain

$$d(u \vee u_1 \vee \dots \vee u_{n-1} \vee v) \leq d(u \vee u_1) + d(u_1 \vee u_2) + \dots +$$
$$+ d(u_{n-1} \vee v) \leq \sum d(a_i) + 2\varepsilon$$

so that

$$d(\bigvee S) < d(u \vee v) + \varepsilon \leq \sum d(a_i) + 3\varepsilon < \sum d(a_i) + \frac{1}{2}\eta$$

in contradiction with (1).

II. Let $a = b$. Choose an arbitrary $c \in S$, $c \neq a$ (obviously S has to have at least two elements). We have

$$d(\bigvee S) > \inf\{\sum_{i=1}^{n} d(a_i) \mid a_1 = a,\ a_n = c,\ a_i \wedge a_{i+1} \neq 0\} + \eta$$

so that, again, there are $a_1 = a, a_2, \dots, a_n = c$ such that

$$d(\bigvee S) > \sum d(a_i) + \frac{1}{2}\eta .$$

We obtain a contradiction

$$d(\bigvee S) < d(u \vee v) + \varepsilon \leq d(u \vee a \vee v) + \varepsilon \leq d(a) + 3\varepsilon \leq$$
$$\leq \sum d(a_i) + 3\varepsilon < \sum d(a_i) + \frac{1}{2}\eta < d(\bigvee S). \quad \square$$

2. Spatial case: metric diameters and pseudometrics

2.1. In this section, a topological space $X = (X, L)$ is given, L is the locale of its open sets. To keep the notation in accord with that of the general case, we will denote the open sets in X by lower case Roman letters. The points of X will be denoted by α, β, γ and δ. If ρ is a pseudometric on X we write

$$\Omega_\rho(\alpha; \varepsilon) = \{\beta \mid \rho(\alpha, \beta) < \varepsilon\}.$$

2.2. Let ρ be a bounded pseudometric on the set X. We construct

$$d: L \to \mathbb{R}_+$$

by putting

(2) $d(x) = \sup\{\rho(\alpha, \beta) \mid \alpha, \beta \in x\}$.

2.3. Proposition: Let the topology of (X, ρ) be weaker than that of X. Then d defined by (2) is a metric diameter.

Proof is a matter of easy checking. Since the sets $\Omega_\rho(\cdot\,;\cdot)$ are open, we can take for u, v in (M) suitable $\Omega(\alpha; \frac{1}{2}\varepsilon)$, $\Omega(\beta; \frac{1}{2}\varepsilon)$. $\square$

2.4. Let $d: L \to \mathbb{R}_+$ be a metric diameter, define

$$\rho : X \times X \to \mathbb{R}_+$$

by putting

(3) $\rho(\alpha, \beta) = \inf\{d(x) \mid \{\alpha, \beta\} \subset x\}$.

2.5. __Proposition:__ The function ρ is a pseudometric on the set X.

__Proof:__ The triangle inequality follows easily from (A), $\rho(\alpha,\alpha) = 0$ from 1.2(iii). Obviously, $\rho(\alpha,\beta) = \rho(\beta,\alpha)$. □

2.6. __Lemma:__ Let ρ be constructed from d by (3). Then

$$\Omega_{\rho}(\alpha;\varepsilon) = \bigvee\{x \mid x \in L,\ \alpha \in x,\ d(x) < \varepsilon\}.$$

Consequently, the topology of (X,ρ) is weaker than L.

__Proof:__ We have

$$\rho(\alpha,\beta) < \varepsilon \text{ iff } \exists x \supset \{\alpha,\beta\},\ x \in L, \text{ such that } d(x) < \varepsilon. \ \square$$

2.7. __Theorem:__ The formulae (2) and (3) constitute a one-one correspondence between the set of all bounded metric diameters d on L and the set of all bounded pseudometrics ρ on X such that the topology of (X,ρ) is weaker than L.

__Proof:__ I. Start with a diameter d, construct ρ by (3) and a new diameter d' from ρ by (2). Obviously,

$$d'(x) \le d(x).$$

Let there be an x and an $\varepsilon > 0$ such that $d(x) > d'(x) + 3\varepsilon$. Take u, v such that $u \wedge x \neq 0 \neq v \wedge x$, $d(u), d(v) < \varepsilon$ and $d(u \vee v) > d(x) - \varepsilon$ (and, hence, $d(u \vee v) > d'(x) + 2\varepsilon$). Choose $\alpha \in u \wedge x$, $\beta \in v \wedge x$. Consider an arbitrary $w \in L$ such that $\{\alpha,\beta\} \subset w$. We have

$$d(u \vee v) \le d(w \vee u \vee v) \le d(w) + 2\varepsilon$$

and hence

$$d(w) \ge d(u \vee v) - 2\varepsilon > d'(x)$$

so that

$$\rho(\alpha,\beta) \ge d(u \vee v) - 2\varepsilon > d'(x)$$

in contradiction with the definition of $d'(x)$.

II. Start with a pseudometric ρ, construct d by (2) and then a new pseudometric ρ' by (3). We obviously have

$$\rho'(\alpha,\beta) \ge \rho(\alpha,\beta).$$

Let $\rho'(\alpha,\beta) > \rho(\alpha,\beta) + 3\varepsilon$. Consider $u = \Omega(\alpha;\frac{1}{2}\varepsilon)$, $v = \Omega(\beta;\frac{1}{2}\varepsilon)$. Thus, $d(u),d(v) < \varepsilon$. Take $\gamma \in u$, $\delta \in v$. We have

$$\rho(\gamma,\delta) \leq \rho(\gamma,\alpha) + \rho(\alpha,\beta) + \rho(\beta,\delta) < \rho(\alpha,\beta) + 2\varepsilon$$

and if $\gamma,\delta \in u$ or $\gamma,\delta \in v$ obviously $\rho(\gamma,\delta) < 2\varepsilon$. Thus

$$d(u \vee v) \leq \rho(\alpha,\beta) + 2\varepsilon < \rho'(\alpha,\beta) - \varepsilon$$

in contradiction with the definition of ρ'. □

2.8. <u>Proposition</u>: (Notation from [6].) Let d be a metric diameter on L, let ρ be obtained by (3). Then

u is open in (X,ρ) iff $u \in L_{\mathcal{U}}$

where $\mathcal{U}$ is the u-basis $\{\{a|d(a) < \varepsilon\} \mid \varepsilon > 0\}$.

<u>Proof</u>: Let u be open in (X,ρ). By 2.6, $u \in L$. Let α be an arbitrary point of u. Take an $\varepsilon > 0$ such that $\Omega(\alpha;2\varepsilon) \subset u$. Put $v = \Omega(\alpha;\varepsilon)$ and consider $A = \{a|d(a) < \varepsilon\}$. We have

$$Av \leq u \text{ and hence } v \overset{\mathcal{U}}{\lhd} u.$$

Since α was arbitrary, $u = \bigvee\{x|x \overset{\mathcal{U}}{\lhd} u\}$.

On the other hand, let $u = \bigvee\{x|x \overset{\mathcal{U}}{\lhd} u\}$. Take an $\alpha \in u$. There is an x, $x \overset{\mathcal{U}}{\lhd} u$ such that $\alpha \in x$ and there has to be an $\varepsilon > 0$ such that, for $A = \{a|d(a) < \varepsilon\}$, $Ax \leq u$. Obviously, $\Omega(\alpha;\varepsilon) \leq Ax$. □

3. <u>Fabricating diameters with stronger properties</u>

3.1. For a star diameter d on a locale L put

$$\delta(x) = \inf_{\varepsilon > 0} \sup\{d(u \vee v) \mid u \wedge x \neq 0 \neq v \wedge x,\ d(u),d(v) < \varepsilon\}.$$

3.2. <u>Lemma</u>: For any $x,y \in L$ we have

$$\delta(x \vee y) \geq d(x \vee y) - d(x) - d(y).$$

<u>Proof</u>: If $x = 0$ or $y = 0$, the right hand side is zero. Thus, we can assume that $x \neq 0 \neq y$.

Let $\sigma(x \vee y) < d(x \vee y) - d(x) - d(y)$. Then we have an $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$,

$$\alpha = \sup\{d(u \vee v) \mid u \wedge (x \vee y) \neq 0 \neq v \wedge (x \vee y),\ d(u), d(v) < \varepsilon\} <$$
$$< d(x \vee y) - d(x) - d(y).$$

Choose u, v such that $u \wedge x \neq 0 \neq v \wedge y$ and $d(u), d(v) < \varepsilon$. We have

$$d(x \vee y) \leq d(x \vee y \vee u \vee v) \leq d(u \vee v) + d(x) + d(y)$$

and we obtain a contradiction

$$\alpha \geq d(u \vee v) \geq d(x \vee y) - d(x) - d(y) > \alpha. \quad \square$$

3.3. <u>Lemma</u>: We have

$$\tfrac{1}{2}d(x) \leq \sigma(x) \leq d(x).$$

Proof: If $u \wedge x \neq 0 \neq v \wedge x$ and $d(u), d(v) < \varepsilon$, we have

$$d(u \vee v) \leq d(x \vee u \vee v) \leq d(x) + 2\varepsilon.$$

Hence, $\sigma(x) \leq d(x)$.

Now, let us have, for some $x \in L$ and $\eta > 0$,

$$\sigma(x) < \tfrac{1}{2}d(x) - \eta.$$

Thus, we have an $\varepsilon > 0$ such that

$$\sup\{d(u \vee v) \mid u \wedge x \neq 0 \neq v \wedge x,\ d(u), d(v) < \varepsilon\} < \tfrac{1}{2}(d(x) - \eta).$$

Take the system $S = \{u \in L \mid d(u) < \varepsilon,\ u \wedge x \neq 0\}$ and choose a fixed $v_0 \in S$. Thus, $x \leq \bigvee\{u \vee v_0 \mid u \in S\}$ and we obtain, by (*),

$$d(x) \leq 2 \sup\{d(u \vee v_0) \mid u \in S\} < d(x) - \eta$$

which is a contradiction. $\square$

3.4. <u>Theorem</u>: For any star-additive diameter d there is a metric diameter σ such that

$$\tfrac{1}{2}d \leq \sigma \leq d.$$

<u>Proof</u>: According to 3.3 it suffices to prove that the σ from 3.1 is a metric diameter. Obviously, it is a prediameter (1.2(ii) is straightforward and 1.2(i) and (iii) follow from 3.3).

(A): Let it not hold. Hence, we have some $a,b \in L$, $a \wedge b \neq 0$, and an $\eta > 0$ such that

$$\delta(a \vee b) > \delta(a) + \delta(b) + \eta .$$

Thus, for a sufficiently small $\varepsilon > 0$,

$$\delta(a \vee b) > \sup \{d(u_1 \vee v_1) \mid u_1 \wedge a \neq 0 \neq v_1 \wedge a,\ d(u_1), d(v_1) < \varepsilon\} +$$
$$+ \sup \{d(u_2 \vee v_2) \mid u_2 \wedge b \neq 0 \neq v_2 \wedge b,\ d(u_2), d(v_2) < \varepsilon\} + \tfrac{1}{2}\eta .$$

Choose u, v, $u \wedge (a \vee b) \neq 0 \neq v \wedge (a \vee b)$, $d(u), d(v) < \varepsilon$ such that

$$\delta(a \vee b) < d(u \vee v) - \frac{1}{2}\eta$$

so that

$$d(u \vee v) > \sup \{d(u_1 \vee v_1) \mid \ldots\} + \sup \{d(u_2 \vee v_2) \mid \ldots\}.$$

Thus, neither $u \wedge a \neq 0 \neq v \wedge a$ nor $u \wedge b \neq 0 \neq v \wedge b$ and we can assume $u \wedge a \neq 0 \neq v \wedge b$. Choose a $w \in L$ such that $d(w) < \varepsilon$ and $w \wedge (a \wedge b) \neq 0$. We obtain a contradiction

$$d(u \vee v) > d(u \vee w) + d(w \vee v) \geq d(u \vee v \vee w).$$

The metric property: By 3.2 we obtain

(4) $\quad \delta(x) \leq \inf_{\varepsilon > 0} \sup \{\delta(u \vee v) + 2\varepsilon \mid u \wedge x \neq 0 \neq v \wedge x,\ d(u), d(v) < \varepsilon\}$.

Let δ be not metric. Then we have an $\varepsilon_0 > 0$ such that for all u, v such that $\delta(u), \delta(v) < \varepsilon_0$ and $u \wedge x \neq 0 \neq v \wedge x$ necessarily

(5) $\quad \delta(u \vee v) \leq \delta(x) - \varepsilon_0$.

Choose an $\varepsilon < \frac{1}{2}\varepsilon_0$. By (4) and 3.3 we have

$$\delta(x) \leq \sup \{\delta(u \vee v) + 2\varepsilon \mid \ldots,\ \delta(u), \delta(v) < \varepsilon\}$$

and hence, using (5), we obtain a contradiction

$$\delta(x) \leq \delta(x) - \varepsilon_0 + 2\varepsilon < \delta(x). \quad \square$$

3.5. Let f be a pre-diameter. For $S \in \mathrm{conn}(L)$ and $a,b \in S$ put $\mu_f(a,b,S) = \inf \{\sum_{i=1}^{n} f(a_i) \mid a_1 = a, a_n = b, a_i \wedge a_{i+1} \neq 0, a_i \in S\}$.

Further, put

$$\mu_f(S) = \sup \{\mu_f(a,b,S) \mid a,b \in S\}.$$

3.6. <u>Observation</u>: 1. Let $b_1 \wedge b_2 \neq 0$. Then

$\mu(a,c,S) \leq \mu(a,b_1,S) + \mu(b_2,c,S)$.

2. Let $S_1 \subset S_2$. Then

$\mu(a,b,S_1) \geq \mu(a,b,S_2)$.

3.7. For $x \in L$ put

$$d_f(x) = \inf\{\mu_f(S) \mid S \in \mathrm{conn}\,(L),\ x \leq \vee S\}.$$

Obviously,

$$d_f \leq f.$$

3.8. <u>Theorem</u>: The function d_f is a star-additive diameter.

<u>Proof</u>: Obviously, d_f is a pre-diameter. Let it not be star-additive. Thus, we have an $S \in \mathcal{S}(L)$ and an $\varepsilon > 0$ such that

(6) $d_f(\vee S) > \sup\{d(a) + d(b) \mid a,b \in S,\ a \neq b\} + 3\varepsilon$.

For each $a \in S$ choose an $S_a \in \mathrm{conn}\,(L)$ such that

$$\vee S_a \geq a \quad \text{and} \quad \mu_f(S_a) < d_f(a) + \varepsilon .$$

Thus, by (6), we have

(7) for any $a,b \in S$, $a \neq b$,
$d(\vee S) > \mu(S_a) + \mu(S_b) + \varepsilon$.

Put $T = \cup\{S_a \mid a \in S\}$. Obviously, $T \in \mathrm{conn}\,(L)$ and $\vee T \geq \vee S$ so that $\mu(T) \geq d(\vee S)$ and hence, by (7),

$$\mu(T) > \mu(S_a) + \mu(S_b) + \varepsilon .$$

Thus, there exist $u,v \in T$ such that

(8) $\mu(u,v,T) > \mu(S_a) + \mu(S_b)$.

We cannot have $u,v \in S_a$ for an a, since then we would have (see 3.6.2) $\mu(u,v,T) \leq \mu(u,v,S_a) \leq \mu(S_a)$. Thus, there are a, b, $a \neq b$, $u \in S_a$ and $v \in S_b$. Choose an $x \in S_a$ and a $y \in S_b$ such that $x \wedge y \neq 0$. Now, (8) and 3.6 yield a contradiction

$$\mu(u,v,T) > \mu(u,x,S_a) + \mu(y,v,S_b) \geq \mu(u,x,T) + \mu(y,v,T) \geq \mu(u,v,T). \quad \square$$

3.9. We will formulate one more condition concerning pre-diameters f:

(3W): for a,b,c such that $a\wedge b \neq 0 \neq b \wedge c$,

$$f(a\vee b\vee c) \leq 2 \max (f(a),f(b),f(c)).$$

<u>Lemma</u>: Let f satisfy (3W). Let $x_1,\dots,x_n$ be such that $x_i \wedge x_{i+1} \neq 0$ for $i = 1,\dots,n-1$. Then

$$f(\bigvee_{i=1}^{n} x_i) \leq 2 \sum_{i=1}^{n} f(x_i).$$

<u>Proof</u> by induction on n. For $n = 1$, $f(x_1) \leq 2f(x_1)$. Let the inequality hold for n, consider $x_1,\dots,x_{n+1}$. Put $\alpha = \sum_{i=1}^{n+1} f(x_i)$ and take the first k such that $\sum_{i=1}^{k} f(x_i) \geq \frac{1}{2}\alpha$. Then

$$\sum_{i=1}^{k-1} f(x_i) < \tfrac{1}{2}\alpha, \qquad \sum_{i=k+1}^{n+1} f(x_i) \leq \tfrac{1}{2}$$

and hence, by the induction hypothesis,

$$f(\bigvee_{i=1}^{k-1} x_i) < \alpha, \qquad f(\bigvee_{i=k+1}^{n+1} x_i) \leq \alpha .$$

Since also $f(x_k) \leq \alpha$ we obtain, using (3W),

$$f(\bigvee_{i=1}^{n+1} x_i) \leq 2\alpha = 2\sum_{i=1}^{n+1} f(x_i). \quad \square$$

3.10. <u>Lemma</u>: Let f be a star diameter satisfying (3W), let μ_f be the function from 3.5. Then for any $S \in \operatorname{conn}(L)$

$$f(\bigvee S) \leq 4\,\mu_f(S).$$

<u>Proof</u>: Fix a $u_0 \in S$ and an $\varepsilon > 0$. For each $u \in S$ choose a sequence $x_1(u),\dots,x_n(u) \in S$ such that $u_0 = x_1(u)$, $u = x_n(u)$, $x_i(u) \wedge x_{i+1}(u) \neq 0$ and

$$\sum f(x_i(u)) < \mu_f(u_0,u,S) + \varepsilon .$$

Put $s(u) = \bigvee x_i(u)$. Evidently, $s(u) \wedge s(v) \geq u_0 \neq 0$ and $u \leq s(u)$ so that

(9) $\quad \bigvee\{s(u) \mid u \in S\} = \bigvee S$ and $\{s(u) \mid u \in S\} \in \mathcal{S}(L)$.

By 3.9 we have

$$f(s(u)) \leq 2\sum f(x_i(u)) < 2\,\mu_f(u_0,u,S) + 2\varepsilon \leq 2\,\mu_f(S) + 2\varepsilon$$

and hence, by $(*)$ and (9),

$$f(\bigvee S) \leq 2 \sup\{f(s(u)) \mid u \in S\} \leq 4\,\mu_f(S) + 4\varepsilon . \quad \square$$

3.11. Theorem: For each star diameter f satisfying (3W) there is a metric diameter d such that

$$\frac{1}{8}f(x) \leq d(x) \leq f(x).$$

Proof: Consider first the function d_f from 3.7. Let S be in conn (L), $x \leq \bigvee S$. By 3.10

$$f(x) \leq f(\bigvee S) \leq 4\,\mu_f(S)$$

and hence $d_f(x) = \inf \{ \mu_f(S) \mid \bigvee S \geq x \} \geq \frac{1}{4}f(x)$.

By 3.8, d_f is star additive so that our statement now follows from 3.4. $\square$

4. (Dia)metrization of uniformities

4.1. A u-basis (resp. wu-basis) $\mathcal{A}$ such that $\tilde{\mathcal{A}} = \mathcal{U}$ (see [6; 3.3, 3.5]) will be referred to as a basis of the uniformity (resp. weak uniformity) $\mathcal{U}$.

It is said to be meet-closed if

$$A,B \in \mathcal{A} \Longrightarrow \exists\, C \in \mathcal{A}\ ,\ C \prec A \wedge B.$$

Obviously, if $\mathcal{A}$ is meet-closed then

$$A \in \mathcal{U} \ \text{ iff } \ \exists\, B \in \mathcal{A}\ ,\ B \prec A.$$

4.2. For a u-basis (wu-basis) $\mathcal{A}$ put

$$m\mathcal{A} = \{ A_1 \wedge \ldots \wedge A_k \mid A_i \in \mathcal{A} \}.$$

By [6; 3.4] we see that $m\mathcal{A}$ is a u-basis (wu-basis) again. Obviously it is meet-closed. Thus, we make an

Observation: If $\mathcal{U}$ has a countable basis, it has a countable meet-closed basis. $\square$

4.3. Lemma: Let a uniformity (resp. a weak uniformity) $\mathcal{U}$ have a countable basis. Then it has a meet closed basis $\mathcal{A} = \{A_0, A_1, \ldots, A_n, \ldots\}$ such that $A_0 = \{e\}$ and, for each n,

$$A_{n+1}^{**} \prec A_n \ (\text{resp. } A_{n+1}^{(2)(2)} \prec A_n).$$

Proof: Take a meet-closed basis $\mathcal{B} = \{B_1, B_2, \ldots, B_n, \ldots\}$

of $\mathcal{U}$. Put $A_0 = \{e\}$, $A_1 = B_1$. Let $A_0,\dots,A_n$ be already defined so that

(α) $A_{k+1}^{**} \prec A_k$ (resp. $A_{k+1}^{(2)(2)} \prec A_k$) for $k<n$,

(β) $A_k \in \mathcal{B}$ for $k \le n$,

(γ) $A_k \prec B_k$ for $k \le n$.

There is a B_r such that $B_r^{**} \prec A_n$ resp. $B_r^{(2)(2)} \prec A_n$ and a $B_s \prec B_r \wedge B_{n+1}$. Put $A_{n+1} = B_s$. □

4.4. <u>Proposition</u>: For each uniformity (resp. weak uniformity) $\mathcal{U}$ there is a system $(\mathcal{U}_i \mid i \in J)$ of uniformities (resp. weak uniformities) with countable bases such that

$$A \in \mathcal{U} \text{ iff } \exists i \ A \in \mathcal{U}_i.$$

Proof: For an $A \in \mathcal{U}$ choose inductively $A_1, A_2, \dots, A_n, \dots$ so that $A = A_1$, $A_{n+1}^* \prec A_n$ (resp. $A_{n+1}^{(2)} \prec A_n$). Put $J = \mathcal{U}$, $\mathcal{A}_A = \{A_i \mid i = 1,2,\dots\}$, $\mathcal{U}_A = \tilde{\mathcal{A}}_A$. □

4.5. For a weak diameter d put

$$\mathcal{U}(d) = \{A \mid \exists\ \varepsilon > 0,\ \{a \mid d(a) < \varepsilon\} \prec A\}.$$

More generally, let $\mathfrak{D}$ be a system of weak diameters. Put

$$\mathcal{U}(\mathfrak{D}) = \tilde{\mathcal{A}} \text{ where } \mathcal{A} = \{\{a \mid d(a) < \varepsilon\} \mid d \in \mathfrak{D},\ \varepsilon > 0\}$$

(using $\tilde{\mathcal{A}}$ has been necessary to ensure the meet property; in the case of one d this is automatic).

Obviously, $\mathcal{U}(d)$, $\mathcal{U}(\mathfrak{D})$ are weak uniformities. If d resp. all the members of $\mathfrak{D}$ are star diameters, $\mathcal{U}(d)$ resp. $\mathcal{U}(\mathfrak{D})$ is a uniformity.

4.6. <u>Theorem</u>: $\mathcal{U}$ is a uniformity with a countable basis iff there is a metric diameter such that $\mathcal{U} = \mathcal{U}(d)$.

(Note that this fact provides the formal definition of metrisability in [3] with a more concrete contents.)

<u>Proof</u>: Consider the basis $A_0, A_1, \dots, A_n, \dots$ from 4.3 and define $f: L \longrightarrow \mathbb{R}_+$ by putting

$$f(x) = \inf\{2^{-n} \mid x \le a \text{ for some } a \in A_n\}.$$

Obviously, f is a pre-diameter. Now, let S be as in $\mathcal{S}(L)$ and let $d(a) \leq 2^{-(n+1)}$ for all $a \in S$. Thus, we have for each $a \in S$ a $b(a) \in A_{n+1}$ such that $b(a) \geq a$. Hence, $\bigvee S \leq \bigvee\{b(a) \mid a \in S\} \in A^*_{n+1} \subset$ $\subset A^{**}_{n+1} \prec A_n$ so that $\bigvee S \leq b$ for some $b \in A_n$. Thus,

$f(a) \leq 2^{-(n+1)}$ for all $a \in S$ implies $f(\bigvee S) \leq 2^{-n}$

and hence $f(\bigvee S) \leq 2 \sup\{f(a) \mid a \in S\}$ so that f is a star diameter.

Now, let x,y,z be such that $x \wedge y \neq 0 \neq y \wedge z$. If $f(x), f(y), f(z) \leq 2^{-(n+1)}$, we have $a,b,c \in A_{n+1}$ such that $x \leq a$, $y \leq b$, $z \leq c$. Hence, $a \vee b \in A^{(2)}_{n+1}$ and $a \vee b \vee c \in A^{(2)(2)}_{n+1} \subset A^{**}_{n+1} \prec A_n$, hence $f(x \vee y \vee z) \leq 2^{-n}$ and we conclude that also (3W) is satisfied. Thus, by 3.11 there is a metric diameter d such that

$$\frac{1}{8} f \leq d \leq f.$$

We check easily that $\mathcal{U} = \mathcal{U}(f)$ and that $\mathcal{U}(f) = \mathcal{U}(d)$.

On the other hand, obviously every $\mathcal{U}(d)$ has the countable basis $\{\{a \mid d(a) < \frac{1}{n}\} \mid n = 1,2,\ldots\}$. □

4.7. <u>Theorem</u>: For every uniformity $\mathcal{U}$ there is a set of metric diameters $\mathcal{D}$ such that $\mathcal{U} = \mathcal{U}(\mathcal{D})$.

<u>Proof</u>: follows easily from 4.4 and 4.6. □

4.8. <u>Remark</u>: The constructions of Section 3 have served the purpose of crossing the gap between the star diameters and the metric ones (of course, this has to be done if we wish to have a generalization of the well-known metrization theorems - see Section 2). To prove just that

$\mathcal{U}$ is a uniformity with a countable basis iff there is a star diameter d such that $\mathcal{U} = \mathcal{U}(d)$

(and a similar weaker analogon of 4.7) one needs the first half of the proof of 4.6 only, without any reference to Section 3.

Similarly, one immediately obtains that

$\mathcal{U}$ is a weak uniformity with a countable basis iff there is a weak diameter d such that $\mathcal{U} = \mathcal{U}(d)$,

and that

For every weak uniformity $\mathcal{U}$ there is a set of weak diameters $\mathfrak{D}$ such that $\mathcal{U} = \mathcal{U}(\mathfrak{D})$.

There seems to be a problem of some interest as to whether the weak diameters in these statements can be replaced by additive ones.

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