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A NONCOMPACT CHOQUET THEOREM

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Abstract: In this paper we are going to establish the validity of Choquet's theorem for a class of noncompact closed convex sets (see also Section 11 in [7]) rich enough to include the class of weakly closed convex sets of Radon probability measures defined on a metrizable topological space X.

Key words and phrases: A bounded convex set, a face of a convex set, the barycenter of a Radon probability measure, simplex.

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1. Introduction. It turns out that the proper tool to link the study of noncompact sets of probability measures with the "compact Choquet's theory" is the concept of Radon measure (see [12] by H. von Weizsäcker or [13] and [14]). In comparison with the Weizsäcker's method we resort to the possibility to embed the space of Radon probability measures into the unit sphere of C*(X) as a face defined by means of Baire measurable affine functions on C*(X) rather than to the possibility "to compactify" the space X, first. We prefer the method to get a representation theorem with Borel and Radon representing measures loosing of course to some extent the generality of the Weizsäcker's results. The present paper provides at the same time both a discussion on the uniqueness of representing measures and an

application to the theory of invariant and ergodic measures.

2. **Fair sets.** All topological spaces treated here are supposed to be Hausdorff. For such a space X denote by $\mathcal{B}(X)$ ($\mathcal{B}_0(X)$) the σ-algebra of Borel (Baire) sets, by $B(X)$ ($B_0(X)$) the space of bounded Borel (Baire) measurable functions on X. A probability measure (p.m.) on $\mathcal{B}(X)$ (Borel probability measure) is called a Radon measure if it is inner regular w.r.t. the paving of compact sets in X. The space of Radon p.m.s' on X will be denoted by $\mathcal{M}(t,1,X)$. Let M be a bounded convex set in a locally convex vector topological space E. Denote by A(M) the set of bounded affine and continuous functions on M and remark that (E^* denotes the dual space to E).

(1) $E^*/M \subset A(M)$, hence the space A(M) separates the points of M.

Having a Borel p.m. P on the set M it follows from (1) that there is at most one point $m \in M$ such that

(2) $a(m) = P(a)$ $(= \int_M a\, dP)$ holds for each $a \in A(m)$.

If this is the case for some m and P, we write $m = b(P)$, call the element $m \in M$ the barycenter of P and say that the p.m. P has its barycenter in M.

The following theorem suggests a simplification in the definition of the barycenter.

Theorem 1. Let M be a relatively compact convex set in a locally convex space E. Then $m \in M$ is the barycenter of a Radon p.m. P on M if and only if

(3) $a(m) = P(a)$ for each $a \in E^*/M$.

Proof. We have to verify that (3) implies (2). Put $Y = \overline{M}$ and define a Radon p.m. Q on the compact convex set Y by

$Q(B) = P(B \cap M),\ B \in \mathcal{B}(Y).$

As the set $E^*/Y + R$ is uniformly dense in $A(Y)$ ([7], p.31), it follows from (3) that

$$(4) \qquad b(Q) = m.$$

Now, consider $a \in A(M)$ and denote by $a^{\downarrow}: Y \longrightarrow R$ ($a^{\uparrow}: Y \longrightarrow R$) the lower (upper) regularization of the function a (see [8], p.98). A simple computation shows that $a^{\downarrow}$ ($a^{\uparrow}$) is a bounded convex lowwe semicontinuous (concave upper semicontinuous) extension of a from M to Y. Thus, it follows from (4) and assertion (a) in [5], p. 274 that

$$a(m) = a^{\downarrow}(m) \leq Q(a\ ) = P(a) = Q(a^{\uparrow}) \leq a^{\uparrow}(m) = a(m)$$

for each $a \in A(M)$, hence $m = b(P)$. Q.E.D.

Our main interest in this section centers around bounded convex sets M in locally convex spaces F which have the property that each point in M is the barycenter of a Radon p.m. on M supported by the set of the extreme points of M, the set which will be denoted by ex(M).

The difficulty in finding measures supported by the extreme points stems, even in the case of a compact set M, from the fact that the set ex(M) need not be a Baire set. Having a noncompact set M we must, moreover, avoid the situations when ex(M) is an empty set. We consult the classical Choquet-Bishop-de Leeuw theorem and suggest to pursue "the compact theory" in the following way: Denote $EX(M) = \{P \in \mathcal{M}(t,1,M): P(F) = 0$ for each G_δ set $F \subset M - ex(M)\}$.

Definition. A bounded convex set $M \subset E$ will be said to be fair (respectively, strongly fair) if for each $m \in M$ there is a measure (respectively, a unique measure) $P \in EX(M)$ such that $m = b(P)$.

Note that each compact set M is fair by Choquet-Bishop-de Leeuw theorem or more precisely by Lemma 4.1 in [7], p. 25 (which states that each element of M may be represented by a maximal Radon p.m. on M) and by Theorem 32,[5], p. 289 (which presents the maximal representing measure as a measure that belongs to EX(M)). Moreover, each compact strongly fair set is a simplex (i.e. compact convex set, each point of which is represented by a unique maximal Radon p.m.) again by Theorem 32 in [5]. On the other hand, an example by Mokobodzky ([7], p. 72) shows that there is a compact simplex which is not strongly fair. Considering the category of bounded convex sets in locally convex spaces, we call its two elements M and Y to be <u>isomorphic</u> if there is an <u>affine homeomorphic bijection</u> (isomorphism) $i: M \longrightarrow Y$. If the set M is isomorphic with a subset of Y via an isomorphism i, we shall write $M \overset{i}{\hookrightarrow} Y$ and call the set M to be a <u>face</u> in Y if, moreover,

$\alpha y_1 = (1-\alpha) y_2 \in i(M),\ \alpha \in (0,1),\ y_1, y_2 \in Y \Longrightarrow y_1, y_2 \in i(M)$.

Now, we collect some obvious properties of fair sets.

<u>Theorem 2</u>. Consider $M \subset E$ a fair set. Then

(a) for each $m \in M$ there is a Radon p.m. P on M such that $m = b(P)$ and $P(B) = 0$ for each $B \subset M - ex(M)$ a Baire set

(b) $M \neq \emptyset \Longrightarrow ex(M) \neq \emptyset$

(c) if a bounded convex set M' is isomorphic with M then M' is a fair set

(d) if the set M is closed in E then $M = \overline{co}\ ex(M)$ (the closed convex hull in E).

Proof. (a) implies (b), directly. Use (1) and the separation theorem to derive (d) from (a). Statement (a) is a consequence to the following simple

Lemma. Every Baire p.m. on an arbitrary topological space is inner regular w.r.t. the paving of Baire G_δ sets. (See [1], p. 195-199.)

Our rather complicated way to express that a representing p.m. is supported by the extreme points may be sometimes simplified in the fashion of Proposition 1.3 in [7], p. 7.

Theorem 3. Let M be a bounded metrizable convex set. Assume that there is a compact convex set Y such that

$M \overset{i}{\hookrightarrow} Y$ and M is a face in Y.

Then ex(M) is a G_δ set in M and we have

$EX(M) = \{P \in \mathcal{M}(t,1,M) : P(ex(M)) = 1\}$.

In particular, the set M is fair (respectively, strongly fair) if and only if for every $m \in M$ there is a Radon p.m. P on M (respectively, there is a unique Radon p.m. P on M) such that $b(P) = m$ and $P(ex(M)) = 1$.

Proof. Without loss of generality assume that $M \subset Y$ and let a metric d to topologize M. Then

$$M - ex(M) = \bigcup_1^\infty F_n, \quad F_n = \{m \in M : m = 2^{-1}(y+z),\ d(y,z) \geq n^{-1},\ y,z \in M\}.$$

It is sufficient to show that the F_n's are closed in M: Let $m_\alpha \in F_n$ be a net tending to some $m \in M$. Then

$$m_\alpha = 2^{-1}(y_\alpha + z_\alpha),\ d(y_\alpha, z_\alpha) \geq n^{-1},\ y_\alpha, z_\alpha \in M.$$

Owing to the compactness of Y the nets $\{y_\alpha\}$, $\{z_\alpha\}$ have cluster points $y \in Y$, $z \in Y$, respectively, such that $m = 2^{-1}(y+z)$. Since M is a face in Y, we may see that y and z are elements of M such that $d(y,z) \geq n^{-1}$. Hence, $m \in F_n$ and set ex(M) is a G_δ set in M. The rest readily follows by Theorem 2(a), given the fact that in the metrizable space M the Baire σ-algebra (the smallest rendering the continuous function measurable) coincides with the Borel σ-algebra. Q.E.D.

A family of noncompact fair sets rich enough to include closed convex sets in the space $\mathcal{M}(t,1,X)$ (for X metrizable) is provided by the following construction:
Denote first

$$Y(\mathcal{K}) = \{y \in Y: \sup\{k(y), k \in \mathcal{K}\} = 1\}$$

for a set Y and a class of functions $\mathcal{K}$ from Y into $[0,1]$.

<u>Theorem 4</u>. Let Y be a compact convex set in a locally convex space E. Consider a class $\mathcal{K}$ of Baire measurable affine and semicontinuous functions $k: Y \to [0,1]$. Then each convex closed set $M \subset Y(\mathcal{K})$ is fair.

Proof. As

$$M = \overline{M}(\mathcal{K}/\overline{M}) \text{ and } \mathcal{B}_0(\overline{M}) \supset \mathcal{B}_0(Y) \cap \overline{M},$$

where $\overline{M}$ denotes the closure of M in Y, we may assume without loss of generality that

(5) $$M = Y(\mathcal{K}).$$

First, we claim that

(6) if Q is a Radon p.m. on Y such that $b(Q) = m \in M$, then there exists $B \in \mathcal{B}_0(Y)$, $B \subset M$ such that $Q(B) = 1$.

Indeed, by (a) in [5], p. 274 and (5)

(7) $$1 = \sup_n k_n(m) = \sup_n Q(k_n) \leq Q(\sup k_n) \leq 1$$

for some sequence $\{k_n\} \subset \mathcal{K}$ since the k_n's are semicontinuous and affine. Thus, putting

$$B = \{y \in Y: \sup k_n(y) = 1\},$$

it follows from (5),(7) by assumption $\mathcal{K} \subset B_0(Y)$ that the set B satisfies the requirements of (6).
It is a simple consequence of (6) that M is a face in Y, hence

(8) $$ex(M) = M \cap ex(Y).$$

To conclude the proof we have to consider $m \in M$ and construct a measure $P \in EX(M)$ such that $m = b(P)$. By Choquet-Bishop-de Leeuw theorem there is a Radon p.m. Q on Y such that

(9) $$m = b(Q) \text{ and } Q(F) = 0 \text{ whenever } F \subset Y-ex(Y) \text{ is a } G_\delta \text{ set.}$$

Putting

$$P(A \cap M) = Q(A) \text{ for each } A \in \mathcal{B}(Y),$$

it follows from (6),(9) and Theorem 1 that P is a Radon p.m. on M such that $m = b(P)$. It remains to show that $P \in EX(M)$. Take a G_δ set $F \subset M-ex(M)$ and write $F = F_1 \cap M$ where $F_1 \subset Y$ is a G_δ set in Y. Furthermore, consider the set B constructed in (6) and $\varepsilon > 0$. By the previous lemma there is a set $F_2 \subset B$ which is G_δ in Y such that $Q(F_2) > 1 - \varepsilon$ holds. It is a consequence of (8) that

$$F_1 \cap F_2 \subset F_1 \cap B \subset M-ex(M) \subset Y-ex(Y).$$

Hence, it follows from (9) that

$$P(F) = Q(F_1 \cap B) < Q(F_1 \cap F_2) + \varepsilon = \varepsilon,$$

since the set $F_1 \cap F_2$ is a G_δ set in Y disjoint from ex Y. Thus, $P(F) = 0$ and $P \in EX(M)$. Q.E.D.

The following theorem is a useful tool to link the concept of a strongly fair set with the concept of a simplex.

Theorem 5. Let M be a fair metrizable convex set. Assume that there is a compact simplex S such that

$$M \overset{i}{\hookrightarrow} S \text{ and } M \text{ is a face in } S.$$

Then the set M is strongly fair. Moreover, the map $m \to P_m$ from M to $EX(M)$ that is established by the relation $m=b(P_m)$ has the following properties:

(i) If $ex(S)$ is closed set then the map $m \to P_m$ is continuous.

(ii) If the topology of M is second countable then the map $m \longrightarrow P_m$ is Borel measurable provided that the topology of M is the relativized weak topology of $\mathcal{M}(t,1,M)$.

Proof. Without loss of generality we assume that $M \subset S$. By Theorem 3 we have

$$EX(M) = \{P \in \mathcal{M}(t,1,M) : P(ex(M)) = 1\}.$$

Put

$$eP(B) = P(B \cap M) \text{ for each } B \in \mathcal{B}(S) \text{ and } P \in EX(M).$$

Note that for $P \in EX(M)$ the Radon p.m. eP is maximal on S by Corollary 9.8 in [7], p. 70. Indeed, if $K \subset S-ex(S)$ is a compact set then

$$eP(K) = P(K \cap M) \leq P(M-ex(M)) = 0,$$

since the set M is a face in S.

Thus, the map

(10) $P \longrightarrow eP$ is an injection from $EX(M)$ to the set of maximal measures on S such that $b(eP)=b(P)$ holds if the measure P has its barycenter in M.

Hence, the set M is strongly fair by (10) as the set S is simplex. Furthermore, denote by $U(M)$ the space of bounded functions on M which are uniformly continuous with respect to the uniformity relativized from S to M. By Theorem 26 in [4], p. 195 and Tietze extension theorem each $f \in U(M)$ may be extended to some $\bar{f} \in C(S)$. As

$$P_m(f) = eP_m(\bar{f}) \text{ for each } f \in U(M) \text{ and } m \in M$$

it follows from (10) and Proposition 9.10 in [7], p. 71 that

(11) $m \longrightarrow P_m(f)$ is a Borel measurable map for $f \in U(M)$,

(12) $m \longrightarrow P_m(f)$ is a continuous map for $f \in U(M)$ provided that $ex(S)$ is a closed set.

Now, it follows by Theorem 8.1 in [9], p. 41 that the sets

(13) $A(P_0,f,\varepsilon) = \{P \in EX(M): |P(f)-P_0(f)| < \varepsilon\}$,

$P_0 \in EX(M)$, $\varepsilon > 0$, $f \in U(M)$,

form a subbase for the weak topology of EX(M). Thus, if ex(S) is a closed set, the map $m \longrightarrow P_m$ is continuous according to (12). If the topology of M is second countable, the same applies for the weak topology of EX(M) (Theorem 11.2 in [11], p. 49) and therefore the Borel σ-algebra in EX(M) is generated by the subbase (13). The map $m \longrightarrow P_m$ is Borel measurable by (11). Q.E.D.

3. Convex sets of Radon measures. Consider a normal topological space X and denote by $\mathcal{M}(t,X)$ the vector space of bounded σ-additive Borel set functions which have its total variation inner regular w.r.t. the paving of compact sets in X. Note that the space $\mathcal{M}(t,X)$ is locally convex when topologized by its usual weak topology, i.e., the coarsest topology for which all maps $m \longrightarrow m(f)$ ($= \int f\, dm$) from $\mathcal{M}(t,X)$ to R are continuous as f varies in the set C(X) of all bounded continuous functions on X.

Furthermore, identify the weak* topologized space $C^*(X)$ with the space of bounded finitely additive regular set functions on the algebra $\mathcal{F}(X)$ generated by the closed sets in X (Riesz theorem, [2], p. 284). Considering the canonical injection i: $\mathcal{M}(t,M) \longrightarrow$ $\longrightarrow C^*(X)$ and putting

(14) $Y = \{m \in C^*(X): m \geq 0,\ m(X) = 1\}$

it is easy to see that

(15) Y is a compact convex set

(Alaoglu theorem, [2], p. 459)

(16) $i: \mathcal{M}(t,1,X) \longrightarrow Y$ is an isomorphism in the sense of Section 2

(Carathéodory theorem and Proposition 1.6.2 in [6], p. 27) and

(17) $i(\mathcal{M}(t,1,X)) = \{m \in Y: \sup\{m(K),\ K \subset X \text{ a compact set}\} = 1\}$.

Now, we are prepared to apply Theorem 4 to get a representation theorem for the space $\mathcal{M}(t,1,X)$.

__Theorem 6.__ Let X be a metrizable space. Then each closed convex set $M \subset \mathcal{M}(t,1,X)$ is fair and it is strongly fair if

(S) $i(M) = i(\mathcal{M}(t,1,X)) \cap S$ for some compact simplex $S \subset C^*(X)$.

Proof. Note that each metrizable space is normal and thus we may employ the setting and notation (14)-(17). Denote by $\mathcal{K}$ the set of all maps $m \longrightarrow m(K)$ from Y into [0,1] where K varies in the set of all compacts in X. Obviously,

$$i(\mathcal{M}(t,1,X)) = Y(\mathcal{K})$$

by (17) where i denotes the canonical isomorphism (16). Now, the closed convex set $i(M) \subset Y(\mathcal{K})$ would be a fair set if only the elements of $\mathcal{K}$ were affine, upper semicontinuous and Baire measurable (according to Theorem 4). We only need to verify that

$K \subset X$, a compact set $\Longrightarrow m \longrightarrow m(K)$ is upper semicontinuous and Baire measurable on Y.

To this end note that $m(K) = \inf m(f_n)$ for all $m \in Y$, where $\{f_n\}$ is a sequence of continuous bounded functions decreasing to the indicator function of K. Thus the map $m \longrightarrow m(K)$ being the infimum of a countable subset of C(Y) is upper semicontinuous and Baire measurable on Y. The set M is fair by Theorem 2(c).

Finally, (S) and (17) imply that M is a face in S. Thus, the set M is strongly fair by Theorem 5. Q.E.D.

In the special case of the space $\mathcal{M}(t,X)$ we may be a little more specific about the properties of the representation suggested by the definition of a fair set. Having a set $M \subset \mathcal{M}(t,1,X)$ we say that a Radon p.m. P on M is a <u>t-Radon p.m.</u> if

$$\sup\{P(T),\ T \in \mathcal{B}(M),\ T \text{ a tight set}\} = 1.$$

Recall that a set $T \subset \mathcal{M}(t,1,X)$ is <u>tight</u> if

for each $\varepsilon > 0$ there is a compact set $K \subset X$ such that $m(K) > 1 - \varepsilon$ for $m \in T$.

Note that there is a family of topological spaces (called Prohorov′spaces) for which the compact subsets of $\mathcal{M}(t,1,X)$ are tight (see [10]). Hence, if X is a Prohorov′s space then each Radon p.m. on $\mathcal{M}(t,1,X)$ is t-Radon.

<u>Theorem 7</u>. Let X be a normal topological space and $M \subset \mathcal{M}(t,1,X)$ a closed convex set. Consider $m_0 \in M$ and a Radon p.m. P on M. Then

(a) $b(P) = m_0 \Longleftrightarrow m_0(f) = \int_M m(f)P(dm),\ f \in C(X)$,

(b) $b(P) = m_0 \Longleftrightarrow m_0(g) = \int_M m(g)P(dm)$ for all $g \in B(X)$.

(c) P has its barycenter in M if and only if it is t-Radon.

Proof. The equivalent definition (a) is a simple consequence of Theorem 1 and Theorem 9 in [2], p. 456, applied to the relatively compact convex set $i(M) \subset Y$ (see (14),(15),(16)). The assertion (b) will be proved for a bounded upper semicontinuous function g, first: We rely on the τ-additivity of the Radon p.m.′s $m_0, m \in M$, P and the equation

$$g = \inf\{f,\ f \in C(X),\ f \geq g\}$$

to get the following more gěneral version of (b):

(18) Let m_o be a τ-additive Borel p.m. on X such that

$$m_o(f) = \int_M m(f)P(dm) \text{ holds for each } f \in C(X).$$

Then

$$m_o(g) = \int_M m(g)P(dm) \text{ for each bounded semicontinuous function } g: X \longrightarrow R.$$

Now, the set of all $g \in B(X)$ for which the relation holds is a linear subspace closed under the sequential bounded convergence, and includes the indicator functions of closed (open) sets, hence all of B(X).

To prove (c) assume first that $b(P) = m_o$ for some $m_o \in M$ and consider a non-decreasing sequence $\{K_n\}$ of compact sets such that $m_o(K_n) \uparrow 1$. It follows from (b) that

$$m_o(K_n) = \int_M m(K_n)P(dm), \quad n \in N$$

and hence $m(K_n) \uparrow 1$ almost surely on M w.r.t. P. Take $\varepsilon > 0$. Applying Egoroff's theorem we obtain a set $T \in \mathcal{B}(M)$ such that $P(T) > 1 - \varepsilon$ and $m(K_n) \uparrow 1$ uniformly for $m \in T$. Thus, the set T is tight and the measure P t-Radon.

On the other hand, consider a t-Radon p.m. P. Using the τ-additivity argument in the same way as in the proof of (b) one proves that the linear functional

$$f \longrightarrow \int_M m(f)P(dm) \text{ (from } C(X) \text{ to } R)$$

satisfies the requirements of Daniell's theorem (see [6], p.66). Hence, there is a regular τ-additive Borel p.m. m_o on X such that

(19) $m_o(f) = \int_M m(f)P(dm)$ holds for $f \in C(X)$.

We prove that m_o is a Radon p.m. . To this end take $\varepsilon > 0$ and a tight set $T \in \mathcal{B}(M)$ such that $P(T) > 1 - \varepsilon$. The tightness of T implies that there is a compact set $K \subset X$ such that $m(K) > 1 - \varepsilon$

for each $m \in T$. It follows from (19) and (18) that

$$m_o(X-K) = \int_M m(X-K)P(dm) \leq 2\varepsilon .$$

Thus, m_o is a regular Borel p.m. such that

$$\sup\{m_o(K),\ K \subset X,\ K \text{ a compact set}\} = 1$$

and therefore a Radon p.m.

Now, as the set M is convex and closed in $\mathcal{M}(t,X)$ we simply apply the separation theorem ([2], p. 452) to verify that $m_o \in M$. Finally, it follows from (19) and (a) that $b(m_o) = P$.

Q.E.D.

It is very simple, now, to summarize our preceding results to get the following representation theorem for a metrizable space:

<u>Theorem 8.</u> (Compare with Theorem 1 in [12].) Let X be a metrizable space and $M \subset \mathcal{M}(t,1,X)$ a nonempty closed convex set. Then

(a) ex(M) is a nonempty G_δ set in M such that $M = \overline{co}\ ex(M)$ (the closed convex hull in $\mathcal{M}(t,1,X)$)

(b) for each $r \in M$ there is a t-Radon p.m. P_r on M such that

(R) $P_r(ex(M)) = 1$ and $r(g) = \int_M m(g)P_r(dm)$, $g \in B(X)$.

Moreover, if the set M is such that (S) in Theorem 6 holds for some compact simplex S, then

(c) for each $r \in M$ there is a unique Radon p.m. P_r on M satisfying requirements (R) and

(d) the map $r \longrightarrow P_r$ from M into EX(M) established by (c) is continuous if ex(S) is a closed set and it is Borel measurable if X is a separable space.

Proof. (a) follows immediately from Theorem 6 and 2. As far as (b),(c),(d) are concerned, let us remark first that the set $M \subset \mathcal{M}(t,1,X)$ inherits its metrizability (separability) from the space X, see Theorem 13 in [11] (Theorem 11.2 in [7], p. 49). Thus, it follows from (17) and the fact that $\mathcal{B}(X) = \mathcal{B}_0(X)$ that we may apply Theorems 6, 3 and 7 to get a verification of (b). Obviously, (c) and (d) are simple consequences to Theorems 6 and 5.

Q.E.D.

The representation (E) in Theorem 8 suggests to consider the vector space $\mathcal{M}(t,M)$ endowed with the B(X)-<u>topology of set wise convergence</u> on the σ-algebra $\mathcal{B}(X)$. Recall that the locally convex B(X)-topology is the topology which makes the set of functionals $m \longrightarrow m(g)$, $g \in B(X)$ to coincide with the space of all B(X)-continuous functionals on $\mathcal{M}(t,X)$ (see [2], p. 453-456). Let $\overline{co}_b A$ denote the closed convex hull of a set A in $\mathcal{M}(t,X)$ endowed with the B(X)-topology while $\overline{co}\, A$ continues to denote the closed convex hull w.r.t. the weak topology of $\mathcal{M}(t,X)$.

<u>Theorem 9</u>. Let X be a metrizable topological space and $M \subset \mathcal{M}(t,1,X)$ a convex set which is weakly closed. Then M is B(X)-closed and $\overline{co}_b ex(M) = M$.

This is an immediate consequence of Theorem 8 (a),(b) and of the separation theorem applied to the set $\overline{co}_b ex(M)$ in the B(X)-topologized space $\mathcal{M}(t,X)$.

4. <u>Ergodic and invariant measures</u>. Let X be a topological space, $\mathcal{T}$ a family of continuous maps $T:X \longrightarrow X$. Denote by $I(\mathcal{T})$ the set of all $\mathcal{T}$-invariant Radon p.m.s on X, by $E(\mathcal{T})$ the set of all ergodic elements of $I(\mathcal{T})$. Recall that a p.m. m is $\mathcal{T}$-<u>invariant</u> if

$$m(T^{-1}B) = m(B) \text{ for } B \in \mathcal{B}(X) \text{ and } T \in \mathcal{T}$$

and that a $\mathcal{T}$-invariant p.m. m is <u>ergodic</u> if

$$m(T^{-1}B \triangle B) = 0,\ T \in \mathcal{T} \Rightarrow m(B) = 0 \text{ or } 1,\ B \in \mathcal{B}(X).$$

<u>Theorem 10</u>. (See also [3] and [12].) Let X be a metrizable space, $\mathcal{T}$ a set of continuous maps $T: X \longrightarrow X$. Then $E(\mathcal{T})$ is a G_δ set in $I(\mathcal{T})$ and to each $r \in I(\mathcal{T})$ there is a unique t-Radon p.m. P_r on $I(\mathcal{T})$ such that

(20) $\quad P_r(E(\mathcal{T})) = 1,\ r(g) = \int_{I(\mathcal{T})} m(g)\,dP_r,\ g \in B(X).$

Moreover,

$$I(\mathcal{T}) \neq \emptyset \Rightarrow E(\mathcal{T}) \neq \emptyset,\ \overline{co}\, E(\mathcal{T}) = I(\mathcal{T})$$

and assuming that X is a separable space, then the map $r \longrightarrow P_r$ established by (20) is Borel measurable.

Proof. It is easy to see that $I(\mathcal{T})$ is a closed convex set in $\mathcal{M}(t,1,X)$ and hence a fair set according to Theorem 6. To see that it is a strongly fair set we shall verify condition (S) of the latter theorem. Define $mT \in C^*(X)$ for $m \in C^*(X)$ and $T \in \mathcal{T}$ by

$$mT(f) = m(f \circ T),\ f \in C(X).$$

Consider a compact convex set (see (14) and (15))

$$S = \{m \in Y : mT = m,\ T \in \mathcal{T}\} \subset C^*(X),$$

the set for which

(21) $\quad i(I(\mathcal{T})) = i(\mathcal{M}(t,1,X)) \cap S$

holds according to (17). To see that S is a simplex, consider the convex cone generated by S, i.e.

$$\mathcal{S} = \{m \in C^*(X) : m \geq 0,\ mT = m \text{ for } T \in \mathcal{T}\},$$

and show that it is a <u>sublattice</u> of the vector lattice $C^*(X)$. Indeed, having $m,n \in S$, $T \in \mathcal{T}$ and $0 \leq f \in C(X)$ we may write by 3.6.6 Corollary in [8], p. 62 that

$$(m \vee n)T(f) \geq \sup\{m(f_1 \circ T) + n(f_2 \circ T),\ 0 \leq f_1,\ f_2 \in C(X),$$

$$f_1 + f_2 = f\xi = (m \vee n)(f).$$

Thus, $(m \vee n)T \geq m \vee n$ and since $(m \vee n)T(1) = m \vee n(1)$ for the constant function 1 we conclude that $(m \vee n)T = m \vee n$. Hence, the set S is a simplex and set $I(\mathcal{T}')$ strongly fair by (21) and Theorem 6.

The rest of our assertion immediately follows from Theorem 2 and Theorem 8, since $ex(I(\mathcal{T}')) = E(\mathcal{T}')$ by Proposition 10.4 in [7], p. 81.

Q.E.D.

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