COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
23,4 (1982)

THE INTERIOR REGULARITY AND THE LIOUVILLE PROPERTY FOR THE QUASILINEAR PARABOLIC SYSTEMS

O. JOHN

Abstract: It is proved that the Liouville property of parabolic quasilinear system - i.e. the fact that each bounded weak solution in R^{n+1} is constant - implies the $C^{0,\alpha}$-regularity of all bounded weak solutions in arbitrary domain. Similar results for quasilinear elliptic systems were established in [3] - [5].

Key words: Quasilinear parabolic system, interior regularity, parabolic Liouville property.

Classification: 35K55

Denote $z = (t,x) = (t,x_1,\dots,x_n) \in R^{n+1}$ and let $u = (u^1,u^2,\dots,u^m)$ be a vector function. We consider the system

$$(1) \quad \frac{\partial u^i}{\partial t} - \frac{\partial}{\partial x_\alpha}\left(a_{ij}^{\alpha\beta}(u)\,\frac{\partial u^j}{\partial x_\beta}\right) = 0, \quad i = 1,\dots,m,$$

which we shall write for the sake of brevity as

$$(2) \qquad u_t - \mathrm{div}_x(A(u)D_x u) = 0.$$

The coefficients $a_{ij}^{\alpha\beta}$ are supposed to be continuous on R^m and

$$(3) \quad (A(u)\eta,\eta) = a_{ij}^{\alpha\beta}(u)\,\eta_\alpha^i\,\eta_\beta^j > 0 \text{ for all } \eta \neq 0,\ u \in R^m.$$

In what follows we shall write for the vector function $u = \{u^i\}_{i=1}^m$ $u \in L_2(Q)$ instead of $u^i \in (L_2Q)$, $i = 1,\dots,m$.

Let $Q \subset R^{n+1}$ be a domain. not necessarily bounded. We say that the function $u \in W^{0,1}_{2,loc}(Q)$ is a weak solution of the system (1) in the domain Q if for each $\varphi \in \mathcal{D}(Q)$ we have

$$\int_Q [u\varphi_t - A(u)D_x \ D_x\varphi]\,dz = 0. \tag{4}$$

(The space $W^{0,1}_{2,loc}(Q)$ is the linear set of all functions u such that u^i and $D_x u^i$ are in $L_{2,loc}(Q)$ for all $i = 1,\dots,m$. On each $Q' \subset\subset Q$, Q' bounded, the seminorm

$$\|u\|_{0,1,Q'} = \|u\|_{L_2(Q')} + \|D_x u\|_{L_2(Q')}$$

can be introduced for all $u \in W^{0,1}_{2,loc}(Q)$.)

The system (1) is said to be regular in a domain Q if each weak solution u of (1) in Q which is bounded belongs to $C^{0,\alpha/2,\alpha}(Q)$.

The space $C^{0,\alpha/2,\alpha}(Q)$ is the linear set of all functions continuous on Q for which on each compact $Q' \subset\subset Q$ the expression

$$\sup\left\{\frac{|u(t,x) - u(t',x')|}{|t - t'|^{\alpha/2} + |x - x'|^{\alpha}};\ (t,x)\in Q',\ (t',x')\in Q', (t,x) \neq (t',x')\right\}$$

is finite.

Finally, we say that the system (1) has parabolic Liouville property if for each weak solution u of (1) in the whole R^{n+1} holds the implication

$$\|u\|_{L_\infty(R^{n+1})} < \infty \implies u \text{ is a constant vector function.} \tag{5}$$

Theorem 1. Let the system (1) have parabolic Liouville property. Then it is regular in each domain $Q \subset R^{n+1}$.

Proof. Denote for $R > 0$, $z_0 \in R^{n+1}$

$$Q(z_0,R) = (t_0 - R^2, t_0 + R^2) \times B(x_0,R), \tag{6}$$

where $B(x_0,R)$ is n-dimensional ball in R^n with the radius R and the center x_0. Denote further by $u_{z_0,R}$ the integral mean value

$$(6') \qquad u_{z_0,R} = \operatorname{mes}^{-1} Q(z_0,R) \int_{Q(z_0,R)} u(z)\,dz.$$

As it was proved in [1], if for the weak solution u of (1) holds in some point $z_0 \in Q$ that

$$(7) \qquad \liminf_{R\to 0_+} \left[R^{-n-2} \int_{Q(z_0,R)} |u(z) - u_{z_0,R}|^2\,dz \right] = 0,$$

then there exists $Q(z_0,\rho)$ such that $u \in C^{0,\alpha/2,\alpha} Q(z_0,\rho)$. (The points for which (7) holds are called the regular points of the weak solution.)

So we want to prove that for each bounded weak solution u of (1) the condition (7) is satisfied in all points $z_0 \in Q$.

Let Q, u and z_0 be fixed, $Q(z_0,R) \subset\subset Q$. Substitute

$$(8) \qquad \tau = \frac{t-t_0}{R^2}, \quad \xi = \frac{x-x_0}{R},$$

$$u_R(\tau,\xi) = u(t_0 + R^2\tau,\ x_0 + R\xi).$$

For an arbitrary constant vector ϕ, we can transform

$$(9) \qquad R^{-n-2} \int_{Q(z_0,R)} |u(z) - u_{z_0,R}|^2\,dz \leqq$$

$$\leqq R^{-n-2} \int_{Q(z_0,R)} |u(z) - \phi|^2\,dz =$$

$$= \int_{Q(0,1)} |u_R(\tau,\xi) - \phi|^2\,dz\,d\xi .$$

(In the first inequality we used the fact that the functional $I(\phi) = \int_{Q(z_0,R)} |u(z) - \phi|^2\,dz$ attains its minimum in the point $\phi = u_{z_0,R}$.)

It is easy to see from (9) and (7) that z_0 is a regular point of u if one can find a subsequence $\{u_{R_n}\}$ $(R_n \to 0)$ of

$\{u_R\}$ such that

(10) $$u_{R_n} \longrightarrow p \text{ in } L_2(Q(0,1)),$$

(11) p is a constant vector function.

To prove (10) and (11) we return to the system (1). Substituting into (4) for t, x and u from (8), we obtain that $u_R(\tau, \xi)$ solves the following system:

(12) $$\int_{(Q)_R} [u_R \varphi_\tau - A(u_R) D_\xi u_R D_\xi \varphi] \, d\tau \, d\xi = 0.$$

Here $(Q)_R$ is the image of Q in the transformation (8).

For $R \to 0+$ $(Q)_R$ expands to the whole R^{n+1}, so that if we choose some fixed $K > 0$, then $Q(0,K) \subset\subset (Q)_R$ for all R smaller than some R(K). So, choosing φ with the support lying in Q(0,K), we can see that each u_R solves the system

(13) $$\int_{Q(0,K)} [u_R \varphi_\tau - A(u_R) D_\xi u_R D_\xi \varphi] \, d\tau \, d\xi = 0,$$

if only $R < R(K)$.

Writing now in (13)

$$A_R(\tau, \xi) = A(u_R(\tau, \xi)), \quad R < R(K)$$

we can see immediately that we can interpret (13) as a class of the linear parabolic systems with the bounded and measurable coefficients. Because of both the estimate

$$\| u_R \|_{L_\infty(Q(0,K))} \leqq \| u \|_{L_\infty(Q)}$$

and the continuity of A(u) we can deduce that the coefficients A_R are equi-bounded and that the corresponding systems have the same constant γ of ellipticity:

$$(A_R(\tau, \xi)\eta, \eta) \geqq \gamma |\eta|^2.$$

(The constant γ as well as the upper bound of $|A_R|$ depend only on $\|u\|_{L_\infty(Q)}$.)

Using the lemmas 4 and 5 from [2] we obtain

$$(14) \qquad |||u_R|||_{W_2^{1/2,1}(Q(0,K/2))} \leqq c\,\|u_R\|_{L_2(Q(0,K))} \leqq$$

$$\leqq c^*(K, \|u\|_{L_\infty(Q(0,K))}),$$

where $W_2^{1/2,1}(Q(0,R))$ is a space of all measurable on $Q(0,R)$ functions w for which the expression $|||w|||_{W_2^{1/2,1}(Q(0,R))} =$

$$= \|w\|_{L_2(Q(0,R))} + \|D_x w\|_{L_2(Q(0,R))} +$$

$$+ \int_{B(0,R)} \int_{-R^2}^{R^2} \int_{-R^2}^{R^2} \frac{|u(t,x) - u(x,s)|^2}{(t-s)^2}\, dt\,ds\,dx$$

is finite.

Because of the compactness of the imbedding of $W_2^{1/2,1}$ into L_2 it follows from (14) that we can choose the subsequence $\{u_n\} = \{u_{R_n}\}$ for which

$$u_n \to p \text{ in } L_2(Q(0,K/2))$$

$$D_x u_n \to D_x p \text{ in } L_2(Q(0,K/2)),$$

$$u_n \to p \text{ almost everywhere in } Q(0,K/2).$$

Using the diagonal method (enlarging $Q(0,K/2)$) we reach the subsequence $\{u_n\} = \{u_{R_n}\}$ of $\{u_R\}$ with the following properties:

$$(16) \qquad \begin{aligned} &u_n \to p \text{ almost everywhere on } R^{n+1}, \\ &u_n \to p \text{ in each } L_2(\Omega),\ \Omega \text{ is bounded in } R^{n+1}, \\ &D_x u_n \to Dp \text{ in each } L_2(\Omega),\ \Omega \text{ is bounded in } R^{n+1}. \end{aligned}$$

From here it follows - after passing to the limit in (12) - that p is a weak solution of (1) in R^{n+1}, so that p is a constant vector function because of Liouville parabolic property.

From (9) we get, putting $\phi = p$ and $R = R_n$, that

$$\lim_{n \to \infty} R^{-n-2} \int_{Q(z_o, R_n)} | u(z) - u_{z_o, R_n} |^2 \, dz = 0.$$

From here it follows immediately (7), q.e.d.

R e f e r e n c e s

[1] M. GIAQUINTA, M. STRUWE: On the partial regularity of weak solutions of nonlinear parabolic systems, Universität Bonn, Preprint No. 455, 1981.

[2] M. GIAQUINTA, E. GIUSTI: Partial regularity for the solutions to nonlinear parabolic systems, Annali di Matematica Pura ed Applicata 97(1973), 253-266.

[3] M. GIAQUINTA, J. NEČAS: On the regularity of weak solutions to nonlinear elliptic systems via Liouville's type property, Comment. Math. Univ. Carolinae 20 (1979), 111-121.

[4] M. GIAQUINTA, J. NEČAS: On the regularity of weak solutions to nonlinear elliptic systems of partial differential equations, J. reine angew. Math. 316(1980), 140-159.

[5] M. GIAQUINTA, J. NEČAS, O. JOHN and J. STARÁ: On the regularity up to the boundary for second order nonlinear elliptic systems, Pacific J. of Math. 99 (1982), 1-17.

Matematicko-fyzikální fakulta, Univerzita Karlova, Sokolovská 83, 18600 Praha 8, Czechoslovakia

(Oblatum 21.6. 1982)