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THE RANGES OF NONLINEAR OPERATORS  
OF THE POLYNOMIAL TYPE  
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**Abstract:** In this paper we prove the existence results for the equation  $Au + Su = f$ , where  $A$  is a polynomial operator on a reflexive Banach space,  $S$  is a strongly continuous nonlinearity.

**Key words:** Polynomial operators, perturbations, strong subasymptote.

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1. **Introduction.** J. Frehse investigated a class of nonlinear functional equations and nonlinear operators of polynomial type (see e.g. [1]). The ranges of these operators are closed linear subspaces with a finite codimension and the equation

$$(1.1) \quad Au = f$$

has at least one solution if  $f$  satisfies the Fredholm condition. Further, J. Frehse deals with the solvability of the equation

$$(1.2) \quad Au + Su = f,$$

where  $S$  is the Landesman-Lazer type nonlinearity (see e.g. [2]).

This paper continues, in some sense, the works [1],[2] and deals with the solvability of the equation (1.2) in section 2, where  $S$  is "subpolynomial-type" nonlinearity. In section 3 the abstract theorems are applied to the examples of polynomial

operators, for example, to the problem

$$\begin{cases} (\Delta - \lambda) [(\Delta u - \lambda u)^5 + (\Delta u - \lambda u)^3] + \\ + |u|^{\sigma} \operatorname{sign} u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

There are also presented results concerning the solvability of (1.2) in section 4, where the operator  $S$  has a vanishing strong subasymptote. For example, there is considered the problem

$$\begin{cases} (\Delta - \lambda) [(\Delta u - \lambda u)^5 + (\Delta u - \lambda u)^3] + \frac{u}{1 + u^2} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

The proof which is published in [5], is analogous to that contained in the papers [3],[4] where equations with linear non-invertible operators in the main part are considered.

**2. Abstract theorems.** We shall investigate continuous maps  $A: B \rightarrow B^*$  where  $B$  is a real reflexive Banach space with a norm  $\|\cdot\|$ ,  $B^*$  is its dual space. We consider following conditions:

(2.1) There exists  $a \geq 0$  such that it holds

- (i) if  $\limsup_{t \rightarrow +\infty} t^{-a} |\langle A(u+tv), v \rangle| < +\infty$   
then  $\langle A(u+tv), v \rangle = \langle Au, v \rangle$  whenever  $t \in \mathbb{R}$ ,  $u, v \in B$ ,
- (ii) if  $\limsup_{t \rightarrow +\infty} t^{-a} |\langle A(tw), v \rangle| < +\infty$   
then  $\langle A(tw), v \rangle = \langle A(0), v \rangle$  whenever  $t \in \mathbb{R}$ ,  $v, w \in B$ .

(2.2) If  $u, v \in B$ ,  $\varphi(t) = \langle A(u+tv), u+tv \rangle$  and

- (i)  $\liminf_{t \rightarrow +\infty} t^{-1} \varphi(t) \geq 0$ ,
  - (ii)  $\limsup_{t \rightarrow +\infty} t^{-1} \varphi(t) < +\infty$ ,
- then  $\lim_{t \rightarrow +\infty} t^{-1} \varphi(t) = 0$ .

Any continuous operator  $A$  satisfying conditions (2.1) and (2.2) will be said  $a$ -polynomial.

An operator  $A$  satisfying

$$(2.3) \quad \liminf_{\|u\| \rightarrow \infty} \|u-v\|^{-1} \langle Au-Av, u-v \rangle \geq 0 \text{ for each } v \in B$$

will be called the asymptotically monotone operator.

(2.4) There exist constants  $K, c > 0$ ,  $p > 1$  and a finite dimensional subspace  $V \subset B$  with a bounded linear projection  $Q: B \rightarrow V$  such that

$$\langle Au, u \rangle \geq c \|u\|^p - K \|Qu\|^p - K \text{ whenever } u \in B.$$

2.5. Definition. A continuous operator  $A: B \rightarrow B^*$  is said regular if the variational inequality

$$\langle Au-f, u-v \rangle \leq 0, \quad v \in K,$$

has a solution  $u \in K$  for any bounded closed convex set  $K \subset B$  and for every  $f \in B^*$ .

The main result of Frehse's work [1] is as follows.

2.6. Theorem. Let  $A: B \rightarrow B^*$  be a regular operator satisfying conditions (2.1)(i) with  $a = 0$ , (2.2)-(2.4) and let  $A(0) = 0$ . Then the equation  $Au = f$  has at least one solution if and only if  $f \perp (R(A))^\perp$ .

Moreover,  $\dim R(A)^\perp \leq \dim V < +\infty$ .

We shall use the next lemma in proofs of the following theorems.

2.7. Lemma. Let  $A: B \rightarrow B^*$  be an asymptotically monotone  $a$ -polynomial operator,  $A(0) = 0$ . Suppose that for some  $v \in B$  there exist constants  $\sigma, C, K \geq 0$  such that the inequality

$$(2.8) \quad \langle Aw, v \rangle \leq C + K \|w\|^\sigma$$

holds for every  $w \in B$ . If  $a \geq \sigma$  then  $v \perp R(A)$ .

Proof. The inequality (2.8) implies  $\langle A(w+tv), v \rangle \leq C + K \|w+tv\|^\sigma$  and from the asymptotical monotonicity of the

operator  $A$  (i.e.  $\liminf_{|t| \rightarrow \infty} |t|^{-1} \langle A(w+tv) - Aw, tv \rangle \geq 0$ ) we obtain  $\langle A(w+tv), v \rangle \geq \langle Aw, v \rangle - \varepsilon$  for every  $t \geq t_0$  with some  $t_0 > 0$ ,  $\varepsilon > 0$ . Together with the supposition (2.8) and the condition (2.1) we have

$$(2.9) \quad \langle A(w+tv), v \rangle = \langle Aw, v \rangle \text{ for every } t \in \mathbb{R}.$$

Using the inequality  $\liminf_{|t| \rightarrow +\infty} |t|^{-1} \langle A(w+tv) - A(2w), -w+tv \rangle \geq 0$  we get that  $\limsup_{|t| \rightarrow +\infty} |t|^{-1} \langle A(w+tv), w \rangle \leq K(w)$  with some constant  $K(w)$ . It yields together with (2.9)  $\limsup_{|t| \rightarrow +\infty} |t|^{-1} \varphi(t) < +\infty$ , where  $\varphi(t) = \langle A(w+tv), w+tv \rangle$ . From conditions (2.2), (2.3),  $A(0) = 0$  it follows that

$$(2.10) \quad \lim_{|t| \rightarrow +\infty} t^{-1} \varphi(t) = 0.$$

Let  $s \in \mathbb{R}$  be fixed. It is obvious that

$$\liminf_{|t| \rightarrow +\infty} |t|^{-1} \langle A(w+tv) - A(sw), (1-s)w+tv \rangle \geq 0$$

and this together with (2.9) yields

$$\liminf_{|t| \rightarrow +\infty} |t|^{-1} [(1-s)\varphi(t) + s \langle Aw, tv \rangle - \langle A(sw), (1-s)w+tv \rangle] \geq 0.$$

According to this fact and with respect to the condition (2.10) we have  $s \langle Aw, v \rangle - \langle A(sw), v \rangle \geq 0$ ,  $-s \langle Aw, v \rangle + \langle A(sw), v \rangle \geq 0$  and

$$(2.11) \quad s \langle Aw, v \rangle = \langle A(sw), v \rangle, \quad s \in \mathbb{R}.$$

If  $a < 1$  then  $0 \leq \sigma < 1$  and as  $s \langle Aw, v \rangle \leq C + K|s|^\sigma \|w\|^\sigma$  we get  $\langle Aw, v \rangle = 0$ , taking the limits  $s \rightarrow \pm \infty$ . This completes the proof for  $a < 1$ .

Let  $a \geq 1$ . There exists  $\nu > 0$  such that  $\|Au\| \leq 1 + \|A(0)\| = 1$  for every  $u \in B$ ,  $\|u\| \leq \nu$ . The inequality

$$\langle Aw, v \rangle = \frac{\|w\|}{\nu} \langle A(\frac{w}{\|w\|} \nu), v \rangle \geq -\frac{\|w\|}{\nu} \|v\|, \quad w \neq 0,$$

is an immediate consequence of (2.11). Therefore, there exists the constant  $L > 0$  such that  $\langle Aw, v \rangle \geq -L \|w\|$ ,  $w \in B$ . Using the inequality (2.8) and the fact that  $a \geq 1$  we obtain

$$\limsup_{t \rightarrow +\infty} t^{-\alpha} |\langle A(tw), v \rangle| < +\infty.$$

From (2.1) we get  $\langle Aw, v \rangle = \langle A(0), v \rangle = 0$ . It means that  $v \perp R(A)$  and the proof of the lemma is complete.

Let  $S: B \rightarrow B^*$  be an operator satisfying conditions

$$(2.12) \quad \|Su\|_{B^*} \leq \alpha + \beta \|u\|^{\sigma}, \quad \alpha, \beta, \sigma \geq 0,$$

(2.13) there exist constants  $G, H > 0$  such that the inequality

$$\liminf_{\|u_j\| \rightarrow +\infty} \|u_j\|^{-1} \langle Su_j - Sw, u_j - w \rangle \geq -G - H \|w\|^{\sigma}$$

is fulfilled for every  $w \in B$ .

2.14. Definition. Let  $V$  be a closed linear subspace of  $B$ ,  $V_r = \{u \in V, \|u\| \leq r\}$ . A mapping  $\Psi: V_1 \rightarrow R$  will be said a strong subasymptote of the operator  $S$  with respect to  $V$  if

$$(2.15) \quad \Psi(z) \leq \liminf_{j \rightarrow +\infty} \langle Su_j, \|u_j\|^{-1}(u_j - w) \rangle, \quad w \in B,$$

holds for any sequence  $\{u_j\}_{j=1}^{+\infty}$  such that  $\|u_j\| \rightarrow +\infty$  and  $\|u_j\|^{-1}u_j \rightarrow z$  (i.e. weakly) for  $j \rightarrow +\infty$ , where  $z \neq 0, z \in V$ .

2.16. Theorem. Let  $A, S: B \rightarrow B^*$  be continuous operators with the following properties

(i)  $A$  is an asymptotically monotone  $\alpha$ -polynomial operator,  $A(0) = 0$  and  $A$  satisfies (2.4),

(ii)  $S$  satisfies (2.12), (2.13) and  $p > 1 + \sigma$ ,  $\alpha \geq \sigma$ ,

(iii)  $A + S$  is a regular operator.

If  $\Psi: (R(A)^\perp)_1 \rightarrow R$  is a strong subasymptote of the operator  $S$  with respect to  $(R(A)^\perp)_1$  and if

$$(2.17) \quad \langle f, z \rangle < \Psi(z) \text{ for every } z \in (R(A)^\perp)_1, z \neq 0,$$

then the equation (1.2) has at least one solution.

Proof. Let us suppose that the equation is not solvable and let  $u_r$  be the solution of the variational inequality

$$(2.18) \quad \langle Au + Su - f, u - w \rangle \leq 0, \quad w \in B_r.$$

Observe that  $u_r \in \partial B_r$  and therefore  $\|u_r\| = r$ . Choose a sequence  $\{r_i\}_{i=1}^{+\infty}$  so that  $\|u_{r_i}\|^{-1} u_{r_i} \rightharpoonup z$  weakly in  $B$ . According to (2.18) with  $w = 0$  and in view of the growth of  $S$  (see (2.12)) we get the inequality  $\langle Au_{r_i}, u_{r_i} \rangle \leq L \|u_{r_i}\|^{1+\sigma}$  for  $i \geq i_0$  with some positive constant  $L$ . Since  $p > 1 + \sigma$  we obtain from (2.4) that  $\liminf_{i \rightarrow +\infty} \|Qu_{r_i}\|^p \|u_{r_i}\|^{-p} \geq \frac{c}{R} > 0$ . The fact that  $\dim R(Q) < +\infty$  implies  $Q(u_{r_i} \|u_{r_i}\|^{-1}) \rightarrow Qz$  in  $B$  for  $i \rightarrow +\infty$  and  $\|Qz\| > 0$ , therefore  $z \neq 0$ .

We claim  $z \perp R(A)$ . Observe that

$$\liminf_{i \rightarrow +\infty} \|u_{r_i}\|^{-1} \langle Au_{r_i} - Aw, u_{r_i} - w \rangle \geq 0,$$

$$\liminf_{i \rightarrow +\infty} \|u_{r_i}\|^{-1} \langle f - Au_{r_i} - Su_{r_i}, u_{r_i} - w \rangle \geq 0$$

and therefore

$$(2.19) \quad \liminf_{i \rightarrow +\infty} \|u_{r_i}\|^{-1} \langle f - Su_{r_i} - Aw, u_{r_i} - w \rangle \geq 0.$$

From (2.13) we have

$$\liminf_{i \rightarrow +\infty} \|u_{r_i}\|^{-1} \langle f - Aw - Sw, u_{r_i} - w \rangle \geq -G - H \|w\|^{\sigma}$$

and this gives the estimate

$$\liminf_{i \rightarrow +\infty} \langle -Aw, u_{r_i} \|u_{r_i}\|^{-1} \rangle \geq -G - H \|w\|^{\sigma} - (\alpha + \beta \|w\|^{\sigma}) - |\langle f, z \rangle|.$$

Consequently,  $\langle Aw, z \rangle \leq G + |\langle f, z \rangle| + \alpha + (\beta + H) \|w\|^{\sigma}$  and the Lemma 2.7 implies  $z \perp R(A)$ .

Observe that the inequality (2.19) yields

$$\langle f, z \rangle - \langle Aw, z \rangle - \liminf_{i \rightarrow +\infty} \langle Su_{r_i}, \|u_{r_i}\|^{-1} (u_{r_i} - w) \rangle \geq 0.$$

As  $\Psi$  is the strong subasymptote of the operator  $S$  we get

$\langle f, z \rangle - \Psi(z) \geq 0$ , which is the contradiction with (2.17) and the proof is complete.

2.20. Proposition. The condition (2.17) is necessary for the solvability of (1.2), if  $\langle Su, z \rangle < \Psi(z)$  for every  $u \in B$ ,  $z \neq 0$ ,  $z \in (R(A)^\perp)_1$ .

Proof. If  $Au + Su = f$  then  $\langle f, z \rangle = \langle Su, z \rangle < \Psi(z)$  for  $z \in (R(A)^\perp)_1$ .

In the case  $\sigma < 1$ , the strong subasymptote of the operator  $S$  can be replaced by more verifiable conditions:

$$(2.21) \liminf_{\|u_i\| \rightarrow +\infty} \|u_i\|^{-1} \langle Su_i - Sw_i, u_i - w_i \rangle \geq -G$$

for every bounded sequence  $\{w_i\}_{i=1}^{+\infty}$ .

(2.22) For every  $z \in (R(A)^\perp)$ ,  $z \neq 0$ , there exist  $t_z \in R$ ,  $v_z \in B$  such that  $\langle S(t_z z + v_z), z \rangle > G$ , where  $G$  is the constant from (2.21).

$$(2.23) \liminf_{i \rightarrow +\infty} \langle S(tz_i + v), -z_i \rangle \leq \langle S(tz + v), -z \rangle$$

holds for any  $t \in R$ ,  $v \in B$  and any sequence  $\{z_i\}_{i=1}^{+\infty} \subset B$ ,  $z_i \rightharpoonup z$  weakly for  $i \rightarrow +\infty$ ,  $z \in (R(A)^\perp)$ ,  $z \neq 0$ .

A strongly continuous operator  $S$  satisfies the condition (2.23).

2.24. Theorem. Let  $A, S: B \rightarrow B^*$  be continuous operators with the following properties

(i)  $A$  is an asymptotically monotone  $\alpha$ -polynomial operator satisfying (2.4),  $A(0) = 0$ ,

(ii)  $S$  satisfies (2.12), (2.21)-(2.23) and  $p > 1 + \sigma$ ,  $\alpha \geq \sigma$ ,  $\sigma < 1$ ,

(iii)  $A + S$  is a regular operator.

Then the equation  $Au + Su = 0$  has at least one solution.

Proof. The condition (2.21) implies (2.13). Let us suppose that the equation  $Au + Su = 0$  is not solvable. Analogously as in the proof of Theorem 2.16 there exists a sequence



$\{u_{r_i}\}_{i=1}^{+\infty}$ ,  $\|u_{r_i}\| \rightarrow +\infty$ ,  $\|u_{r_i}\|^{-1} u_{r_i} \rightharpoonup z$  weakly in  $B$  for  $i \rightarrow +\infty$ ,  $z \in R(A)^\perp$ ,  $z \neq 0$ , and  $\langle Au_{r_i} + Su_{r_i}, u_{r_i} - w \rangle \leq 0$  for every  $w \in B_{r_i}$ . As the operator  $S$  satisfies (2.21) and (2.22) we have

$$\begin{aligned} -G &\leq \liminf_{i \rightarrow +\infty} \|u_{r_i}\|^{-1} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z) - Su_{r_i}, \\ &\quad t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z - u_{r_i} \rangle = \\ &= \liminf_{i \rightarrow +\infty} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z) - Su_{r_i}, -u_{r_i} \|u_{r_i}\|^{-1} \rangle \\ &\text{because } \sigma < 1. \text{ The operator } A + S \text{ is regular and therefore we} \\ &\text{get } \langle Au_{r_i} + Su_{r_i}, -u_{r_i} \rangle \geq 0 \text{ and} \\ &\liminf_{i \rightarrow +\infty} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z) + Au_{r_i}, -u_{r_i} \|u_{r_i}\|^{-1} \rangle \geq -G. \end{aligned}$$

Further,  $A$  is asymptotically monotone, e.g.

$$\liminf_{i \rightarrow +\infty} \langle -Au_{r_i}, -u_{r_i} \|u_{r_i}\|^{-1} \rangle \geq 0$$

and

$$\liminf_{i \rightarrow +\infty} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z), -u_{r_i} \|u_{r_i}\|^{-1} \rangle \geq -G.$$

From (2.23) we obtain  $\langle S(t_z z + v_z), z \rangle \leq G$ , which is the contradiction with (2.22).

3. Examples. Let  $P_j: \mathbb{R}^s \rightarrow \mathbb{R}$ ,  $j = 1, 2, \dots, s$ , be polynomials satisfying the following conditions (with  $C, K, c > 0$ )

$$(3.1) \quad |P_j(\xi)| \leq C(1 + |\xi|^{p-1}) \text{ for every } \xi \in \mathbb{R}^s,$$

$$(3.2) \quad \sum_{j=1}^s P_j(\xi) \xi_j \geq c |\xi|^p - K \text{ for every } \xi \in \mathbb{R}^s,$$

$$(3.3) \quad \sum_{j=1}^s (P_j(\xi) - P_j(\eta)) (\xi_j - \eta_j) \geq 0 \text{ for all } \xi, \eta \in \mathbb{R}^s.$$

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with a smooth boundary and let  $V = W^{2m, p}(\Omega) \cap W_0^m, p(\Omega)$ ,  $p > 1$ . We define

$L_j u = \sum_{|\kappa|, |q| \leq m} (-1)^r D^r (a_{r,q}^{(j)}(x) D^q u)$ ,  $j = 1, \dots, s$ ,  
 for every  $u \in V$  where  $a_{r,q}^{(j)} \in C^\infty(\bar{\Omega})$  ( $|\kappa|, |q| \leq m$ ,  $j = 1, \dots, s$ ). Let

$$\sum_{|\kappa|, |q| \leq m} (-1)^m a_{r,q}^{(j)}(x) \xi^{r+q} \geq \alpha |\xi|^{2m}, \quad j = 1, \dots, s,$$

hold with some  $\alpha > 0$  for every  $\xi \in \mathbb{R}^N$ . Let us define the operator  $A: V \rightarrow V^*$  by

$$\langle Au, v \rangle = \sum_{j=1}^s \int_{\Omega} P_j(L_1 u, \dots, L_s u) L_j v, \quad v \in V.$$

Using the Theorem 2.6, we see that the equation  $Au = f$  is solvable if  $(f - A(0)) \perp (R(A) - A(0))^\perp$ . Let us remark that for  $s = 1$  it is possible to show: if we consider the operator  $A: V/\text{Ker}[L_1] \rightarrow (V/\text{Ker}[L_1])^*$  then this result follows from the theory of monotone operators and  $(R(A) - A(0))^\perp = \text{Ker}[L_1]$ .

Let the function  $\varphi$  be continuous, odd, increasing,  $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$  and  $|\varphi(t)| \leq \bar{\alpha} + \bar{\beta} |t|^\sigma$ ,  $t \in \mathbb{R}$ , with some  $\bar{\alpha}, \bar{\beta}, \sigma > 0$ . Let  $2mp > N$ . We define the operator  $S: V \rightarrow V^*$  by

$$\langle Su, v \rangle = \int_{\Omega} \varphi(u) v, \quad v \in V.$$

We note that the inequality (2.12) holds with some constants  $\alpha, \beta$ . Let us assume the conditions

$$(3.4) \quad \limsup_{t \rightarrow +\infty} \varphi(\omega t) [\varphi(t)]^{-1} = \chi(\omega) < +\infty$$

for every  $\omega \geq 1$ , where  $\chi$  is a continuous function with

$$\lim_{\omega \rightarrow 1^+} \chi(\omega) = 1,$$

$$(3.5) \quad \text{meas } \Omega > 2 \text{ meas } \{x \in \Omega; z(x) = 0\}$$

for every  $z \in (R(A) - A(0))^\perp$ ,  $z \neq 0$ .

**3.6. Proposition.** The mapping  $\Psi: ((R(A) - A(0))^\perp)_1 \rightarrow \{K\}$ ,

where  $K$  is a real number, is a strong subasymptote of the

operator  $S$  defined above with respect to  $(R(A) - A(0))^\perp$ .

**Proof.** We assume that  $A(0) = 0$  and that for a sequence  $\{u_n\}_{n=1}^{+\infty} \subset V$  it is  $\|u_n\|^{-1} \rightarrow +\infty$ ,  $(u_n - w) \|u_n\|^{-1} \rightarrow z$  weakly for  $n \rightarrow +\infty$ ,  $z \in R(A)^\perp$ ,  $z \neq 0$ ,  $w \in V$ . It suffices to show that

$$\liminf_{n \rightarrow +\infty} \int_{\Omega} \varphi(u_n) \frac{u_n - w}{\|u_n\|} - K \geq 0.$$

As  $W^{2m,p}(\Omega)$  is compactly imbedded into  $C(\bar{\Omega})$  we have  $u_n \|u_n\|^{-1} \rightarrow z$  and  $(u_n - w) \|u_n\|^{-1} \rightarrow z$  in  $L_\infty(\Omega)$ . If we denote  $\Omega_\varepsilon^+ = \{x \in \Omega; z(x) \geq \varepsilon\}$ ,  $\Omega_\varepsilon^- = \{x \in \Omega; z(x) \leq -\varepsilon\}$ ,  $\Omega_\varepsilon = \Omega_\varepsilon^- \cup \Omega_\varepsilon^+$  then according to (3.4) and (3.5) there exist  $\varepsilon > 0$ , an integer  $k_0 > 1$  such that the inequality

$$(3.6) \quad \text{meas } \Omega_\varepsilon - \frac{k+1}{k-1} \chi\left(\frac{k+1}{k-1}\right) \text{meas}(\Omega \setminus \Omega_\varepsilon) > 0$$

holds for every  $k \geq k_0$ . There exists a natural number  $n_0$  such that

$$z(x) - \frac{\varepsilon}{k_0} \leq \frac{u_n(x) - w(x)}{\|u_n\|} \leq \frac{\varepsilon}{k_0} + z(x) \quad \text{a.e. in } \Omega,$$

$$z(x) - \frac{\varepsilon}{k_0} \leq \frac{u_n(x)}{\|u_n\|} \leq \frac{\varepsilon}{k_0} + z(x) \quad \text{a.e. in } \Omega$$

for every  $n \geq n_0$ . So we get

$$\begin{aligned} \int_{\Omega} \varphi(u_n) \frac{u_n - w}{\|u_n\|} &\geq \int_{\Omega_\varepsilon^+} \varphi(u_n) \frac{u_n - w}{\|u_n\|} + \int_{\Omega_\varepsilon^-} \varphi(-u_n) \frac{-u_n + w}{\|u_n\|} - \\ &- \int_{\Omega \setminus \Omega_\varepsilon} \varepsilon \frac{k_0 + 1}{k_0} \varphi\left(\varepsilon \frac{k_0 + 1}{k_0} \|u_n\|\right) \geq \\ &\geq \int_{\Omega_\varepsilon} \varepsilon \frac{k_0 - 1}{k_0} \varphi\left(\varepsilon \frac{k_0 - 1}{k_0} \|u_n\|\right) - \int_{\Omega \setminus \Omega_\varepsilon} \varepsilon \frac{k_0 + 1}{k_0} \varphi\left(\varepsilon \frac{k_0 + 1}{k_0} \|u_n\|\right) \geq \\ &\geq \varepsilon \frac{k_0 - 1}{k_0} \varphi\left(\varepsilon \frac{k_0 - 1}{k_0} \|u_n\|\right) \text{meas } \Omega_\varepsilon - \end{aligned}$$

$$- \frac{k_0 + 1}{k_0} \varepsilon \left[ \chi \left( \frac{k_0 + 1}{k_0 - 1} \right) + \psi_n \right] \text{meas}(\Omega \setminus \Omega_\varepsilon) \varphi \left( \varepsilon \frac{k_0 - 1}{k_0} \|u_n\| \right),$$

where  $\psi_n \rightarrow 0$  for  $n \rightarrow +\infty$ . Observe that

$$\int_\Omega \varphi(u_n) \frac{u_n - w}{\|u_n\|} \geq \varepsilon \frac{k_0 - 1}{k_0} \varphi \left( \varepsilon \frac{k_0 - 1}{k_0} \|u_n\| \right) [\text{meas } \Omega_\varepsilon - \\ - \frac{k_0 + 1}{k_0 - 1} (\chi \left( \frac{k_0 + 1}{k_0 - 1} \right) + \psi_n) \text{meas}(\Omega \setminus \Omega_\varepsilon)].$$

Denote the expression in the square brackets by  $c_n$ . It follows from (3.6) that  $\lim_{n \rightarrow +\infty} c_n > 0$  and therefore

$$\lim_{n \rightarrow +\infty} \varepsilon \frac{k_0 - 1}{k_0} \varphi \left( \varepsilon \frac{k_0 - 1}{k_0} \|u_n\| \right) c_n = +\infty.$$

The proof is finished.

If the operator  $A$  satisfies the condition (3.5) then the Theorem 2.16 can be applied. If  $\sigma < 1$  then the operator  $S - f$  satisfies the conditions (2.21)-(2.23) and the Theorem 2.24 can be used. In these cases, if  $p > 1 + \sigma$ , a  $\delta > 0$  then the equation  $Au + Su = f$  has at least one solution.

For example, the problem

$$(\Delta - \lambda)[(\Delta u - \lambda u)^5 + (\Delta u - \lambda u)^3] + |u|^{\sigma'} \text{sign } u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega$$

has at least one weak solution  $u \in W_0^{1,6}(\Omega) \cap W^{2,6}(\Omega)$  for  $0 < \sigma < 3$ .

**4. Problems with a bounded nonlinearity.** Let  $B$  be a linear closed subspace of  $W^{k,p}(\Omega)$ ,  $kp > N$ ,  $p > 1$ ,  $A(0) = 0$ ,

$$(4.1) \quad \langle Su, v \rangle = \int_\Omega \varphi(u) v, \text{ for } u, v \in B,$$

where the function  $\varphi$  is continuous, odd,  $\lim_{|t| \rightarrow +\infty} \varphi(t) = 0$ . Then

$\|Su\|_{B^*} \leq \varphi$  for every  $u \in B$  with some constant  $\varphi$ . Further,

we shall assume the following conditions be satisfied

(4.2) for all  $w \in R(A)^\perp$ ,  $t \in R$ ,  $v \in B$  it is  $A(v + tw) = Av$ ,

(4.3) there exists a bounded linear projection  $Q: B \rightarrow R(A)^\perp$  and  $\langle Au, u \rangle \geq C \|u\|^p - K \|Qu\|^p - L$  for every  $u \in B$ , where  $p > 1$ ,  $C, K, L > 0$ .

**4.4. Proposition.** Let the function  $t \mapsto \langle A(u + tv), w \rangle$  be a polynomial for any fixed  $u, v, w \in B$ . If  $A$  is regular and satisfies (2.3), (2.4),  $A(0) = 0$ , then the condition (4.2) is fulfilled.

The proof can be found in Frehse's papers or in [5].

Let  $\Psi: (0, +\infty) \rightarrow (0, +\infty)$  be the increasing function satisfying

$$\sup_{\substack{w \in R(A)^\perp \\ \|w\|_{C(\bar{\Omega})} = 1}} \int_{\Omega_\varepsilon(w)} |w| \leq \Psi(\varepsilon),$$

where  $\Omega_\varepsilon(w) = \{x \in \Omega; 0 < |w(x)| < \varepsilon\}$  and such that

$$\limsup_{\varepsilon \rightarrow 0^+} [\Psi(\varepsilon)]^{-1} \Psi(\omega\varepsilon) < +\infty \text{ for every } \omega \in (0, +\infty).$$

**4.5. Theorem.** Let a regular asymptotically monotone 0-polynomial operator  $A$  satisfy the conditions (4.2), (4.3),  $A(0) = 0$  and let  $S$  be given by (4.1). If

$$(4.6) \quad \lim_{\xi \rightarrow +\infty} [\Psi(\frac{1}{\xi})]^{-1} \min_{\tau \in \langle a, \xi \rangle} \varphi(\tau) = +\infty$$

for some  $a > 0$  then the equation  $Au + Su = f$  has at least one solution for an arbitrary  $f \perp R(A)^\perp$ .

Sketch of the proof. Let us consider the function

$$\tilde{\varphi}: \xi \mapsto \begin{cases} \varphi(\xi) & \text{for } |\xi| \leq b, \\ \varphi(b) & \text{for } \xi > b, \\ \varphi(-b) & \text{for } \xi < -b, \end{cases}$$

and the corresponding equation  $Au + \tilde{S}u = f$ . From the Theorem 2.16 this equation has a solution  $u$  because

$$0 = \sup_{\substack{w \in R(A)^\perp \\ \|w\|_{C(\bar{\Omega})} = 1}} |\langle f, w \rangle| < |\tilde{\varphi}(b)| \inf_{\substack{w \in R(A)^\perp \\ \|w\|_{C(\bar{\Omega})} = 1}} \int_{\Omega} |w|.$$

Using the condition (4.2) we can obtain a priori estimate

$$\|Q^c u\|_{C(\bar{\Omega})} \leq c_1 = c_1(\|f\|_{B^*}).$$

Further, methods from [3],[4] give a priori estimate

$$\|Qu\|_{C(\bar{\Omega})} \leq c_3 = c_3(a, \tilde{\varphi}, f),$$

where  $a > 0$ ,

$$c_3 = \frac{a + c_1}{\Psi^{-1}(c_2(\inf_{\xi \geq a} \tilde{\varphi}(\xi) + \sup_{\xi \in R} |\tilde{\varphi}(\xi)|)^{-1})},$$

$$c_2 = c_2(a, \tilde{\varphi}, f) = \inf_{\substack{w \in R(A)^\perp \\ \|w\|_{C(\bar{\Omega})} = 1}} (\inf_{\xi \geq a} \tilde{\varphi}(\xi) \int_{\Omega} |w|).$$

If there exist numbers  $a, b \in R$ ,  $0 < a < b$ , such that  $b > c_1(\tilde{\varphi}, f) + c_3(a, \tilde{\varphi}, f)$  then the solution  $u$  of the equation  $Au + \tilde{S}u = f$  is also the solution of the equation  $Au + Su = f$  because  $\tilde{S}u = Su$ . The condition (4.6) guarantees the existence of such numbers  $a, b$ .

For example, the problem

$$\begin{cases} (\Delta - \lambda) [(\Delta u - \lambda u)^5 + (\Delta u - \lambda u)^3] + \frac{u}{1+u^2} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

has at least one weak solution  $u \in W_0^{1,6}(\Omega) \cap W^{2,6}(\Omega)$  if  $f \perp \text{Ker}[\Delta - \lambda \text{id}]$ .

It is also possible to apply the abstract results to the existence of solution of the Neuman problem

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} [(c + |\nabla u|^2)^{\frac{p}{2}} - 1] \frac{\partial u}{\partial x_i} + \frac{u}{1 + |u|^k} = f \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where  $c > 0$ ,  $p > 1$ ,  $k \geq 2$ . If  $f \in L_1(\Omega)$ ,  $\int_{\Omega} f(x) dx = 0$ , this problem has at least one weak solution  $u \in W^{1,p}(\Omega)$ .

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