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EULER POLYGONS AND KNESER'S THEOREM FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract: By (PC) we denote the problem of finding the solution of the differential equation $x' = f(t,x)$ satisfying the initial condition $x(0) = x_0$, where t belongs to a compact real interval and f is a function with values in a Banach space E. In this note we are interested in the study of the problem (PC) with applying the method of Euler polygons. Using this method we obtain some Kneser-Szufla type results for (PC) (the set of all solutions of the problem(PC) is a nonempty continuum in the space C(J,E)) when the function f satisfying regularity Ambrosetti type condition with respect to the "measure of noncompactness α ".

Key words: Differential equations with values in Banach space, Euler polygons, structure of the set solutions, measure of noncompactness.

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1. Introduction and notations. Throughout this paper we assume that $I = [0,a]$, E is a Banach space with the norm $\|\cdot\|$, $B = \{x \in E: \|x - x_0\| \le r\}$, f is a uniformly continuous function from $I \times B$ into E, and $M = \sup\{\|f(t,x)\| : (t,x) \in I \times B\} < \infty$. Moreover, let $J = [0,h]$ where $h = \min(a, M^{-1}r)$.

By (PC) we shall denote the problem of finding the solution of the differential equation

$$x' = f(t,x)$$

satisfying the initial condition $x(0) = x_0$.

In this note we are interested in the study of the problem (PC) with applying the method of Euler polygons. More precisely, using this method we obtain some Kneser-Szufla ([9]) type results for (PC) (the set of all solutions of the problem (PC) is a nonempty continuum in the space C(J,E)) when the function f satisfying regularity Ambrosetti type condition (see [1],[3]) with respect to the "measure of noncompactness α". The idea of our work is closed in [10]. See also [5] - [8].

2. <u>Definitions</u>. Denote by C(J,E) the space of all continuous functions from J to E, with the usual supremum norm $|||\cdot|||$.

<u>Definition 1</u>. A function $x:J \to E$ is said to be a solution of the problem (PC) on the interval J, if it is a differentiable on J such that $x(0) = x_0$, $x(t) \in B$ for t in J, and $x'(t) = f(t,x(t))$ on J. Moreover, denote by S the set of all solutions of (PC) on J.

<u>Definition 2</u>. Let $0 < \varepsilon \leq h$, $0 \leq p \leq h$ and let $v:J \to B$ be a function such that $v(0) = x_0$ and $\| v(p) - x_0 \| \leq Mp$. We will call an (ε,p,v)-polygon Euler line for (PC) on J any function $y(\cdot;\varepsilon,v)$ of the form:

$$y(t;\varepsilon,p,v) = \begin{cases} v(t) \text{ for } 0 \leq t \leq p; \\ v(p) \text{ for } p \leq t \leq p+\varepsilon; \\ y(t_i;\varepsilon,p,v) + \\ \quad + (t-t_i)f(t_i;y(t_i;\varepsilon,p,v)) \\ \quad \text{for } t_i \leq t \leq t_{i+1}, \end{cases}$$

here (without loss of generality) we assume that $r_o = p/\varepsilon$ and $r^o = h/\varepsilon$ are positive integers, $r^o > 1$ and $t_i = i\varepsilon$ for $i = r_o + 1, r_o + 2, \dots, r^o - 1$.

<u>Definition 3</u>. By an ε-polygon Euler line of the problem (PC) we shall call any (ε,p,v)-polygon Euler line of (PC) with $p = 0$ and $v(t) \equiv x_0$ on J.

<u>Definition 4</u>. Let n be a positive integer. By S_n we can

denote the set of all $\frac{1}{n}$ -approximate solutions of the problem (PC) on the interval J. Here, a function $u:J \longrightarrow E$ is said to be $\frac{1}{n}$ -approximate solution of (PC) on J, if it satisfies the following conditions:

(i) $u(0) = x_0$ and $\| u(t'') - u(t') \| \leq M \, | t'' - t' |$ for t', t'' in J;

(ii) $\| u(t'') - u(t') - \int_{t'}^{t''} f(s,u(s))ds \| \leq n^{-1} \, | t'' - t' |$ for all $0 \leq t' \leq t'' \leq h$;

(iii) $\sup_{t \in J} \| u(t) - x_0 - \int_0^t f(s,u(s))ds \| < 1/n$.

Definition 5. We say that the function f satisfies the condition (s) if any set $\{u_n : n = 1,2,\dots\}$ with u_n in $\overline{S_n}$ (= the closure of S_n in C(J,E)) is a conditionally compact subset of C(J,E).

Let S_0 be the set of all solutions of (PC) which are a limit of uniformly convergent sequence of Euler polygonal lines which are approximate solutions of this problem on J. It can be demonstrated (cf. [5]) that under suitable assumptions S_0 is a nonempty continuum in the space C(J,E). Note that $S \neq S_0$.

Indeed, let $f(t,x) = \sqrt{x}$ for $t \geq 0$ and $x \geq 0$. Let us put $\varphi_0(t) = 0$ for $t \geq 0$, and

$$\varphi_\xi(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \xi , \\ (t-\xi)^2/4 & \text{for } t > \xi \end{cases}$$

where $\xi > 0$. It is easy to prove that φ_0 and φ_ξ ($\xi > 0$) are solutions of (PC) with $x_0 = 0$. Moreover, $\varphi_0 \in S_0$ and $\varphi_\xi \notin S_0$ for each ξ.

3. Some properties. First we prove that $S = \bigcap_{n=1}^{\infty} \overline{S_n}$. Obviously $S \subset \bigcap_{n=1}^{\infty} S_n$. Let $|||u_i - u||| \longrightarrow 0$ with $u_i \in S_n$ for all i. Since f is uniformly continuous and

$$\| u(t) - x_0 - \int_0^t f(s,u(s))ds \| \le |||u_i - u||| + 1/n +$$
$$+ \int_0^t \| f(s,u_i(s)) - f(s,u(s))ds,$$

so letting $i \to \infty$, we obtain

$$\| u(t) - x_0 - \int_0^t f(s,u(s))ds \| \le 1/n$$

for t in J. This implies $\bigcap_{n=1}^{\infty} \overline{S_n} \subset S$ and we are done.

Let $y(\cdot; \varepsilon,p,v)$ be an (ε,p,v)-polygon Euler line for (PC) on J and let $t_i \le t \le t_{i+1}$ (here $t_i = i\varepsilon$ for $i = r_0 + 1, r_0 + 2, \ldots, r^0 - 1$). We have

$$(1) \quad y(t; \varepsilon,p,v) = v(p) +$$
$$+ \sum_{m=1}^{i-r_0-1} (t_{r_0+m+1} - t_{r_0+m}) f(t_{r_0+m}, y(t_{r_0+m}; \varepsilon,p,v)) +$$
$$+ (t - t_i) f(t_i, y(t_i; \varepsilon,p,v))$$

and

$$(2) \quad \| y(t; \varepsilon,p,v) - x_0 - \int_0^t f(s,y(s; \varepsilon,p,v))ds \| \le$$
$$\le \| v(p) - x_0 - \int_0^{r} f(s,v(s))ds \| +$$
$$+ \int_{r}^{t_{r_0+1}} \| f(s,y(s; \varepsilon,p,v)) \| ds + I_0 \le$$
$$\le \sup_{t \in J} \| v(t) - x_0 - \int_0^t f(s,v(s))ds \| + M\varepsilon + I_0,$$

where

$$I_0 = \sum_{m=1}^{i-r_0-1} \int_{t_{r_0+m}}^{t_{r_0+m+1}} \| f(t_{r_0+m}, y(t_{r_0+m}; \varepsilon,p,v)) -$$
$$- f(s,y(s; \varepsilon,p,v)) \| ds +$$
$$+ \int_{t_i}^{t} \| f(t_i, y(t_i; \varepsilon,p,v)) - f(s,y(s; \varepsilon,p,v)) \| ds.$$

Hence, for $t_j \le t' \le t_{j+1}$ and $t_k \le t'' \le t_{k+1}$ with $j \le k$,

$$(3) \quad \| y(t''; \varepsilon,p,v) - y(t'; \varepsilon,p,v) - \int_{t'}^{t''} f(s,y(s; \varepsilon,p,v))ds \| =$$
$$= \| (t_{j+1} - t') f(t_j, y(t_j; \varepsilon,p,v)) +$$

$$+ \sum_{m=1}^{k-j-1} (t_{j+m+1} - t_{j+m}) f(t_{j+m}, y(t_{j+m}; \varepsilon, p, v)) +$$

$$+ (t'' - t_k) f(t_k, y(t_k; \varepsilon, p, v)) -$$

$$- \int_{t'}^{t_{j+1}} f(s, y(s; \varepsilon p, v)) ds -$$

$$- \sum_{m=1}^{k-j-1} \int_{t_{j+m}}^{t_{j+m+1}} f(s, y(s; \varepsilon, p, v)) ds -$$

$$- \int_{t_k}^{t''} f(s, y(s; \varepsilon, p, v)) ds \| \leq$$

$$\leq \int_{t'}^{t_{j+1}} \| f(t_j, y(t_j; \varepsilon, p, v)) - f(s, y(s; \varepsilon, p, v)) \| ds +$$

$$+ \sum_{m=1}^{k-j-1} \int_{t_{j+m}}^{t_{j+m+1}} \| f(t_{j+m}, y(t_{j+m}; \varepsilon, p, v)) -$$

$$- f(s, y(s; \varepsilon, p, v)) \| ds +$$

$$+ \int_{t_k}^{t''} \| f(t_k, y(t_k; \varepsilon, p, v)) - f(s, y(s; \varepsilon, p, v)) \| ds.$$

Moreover, it can be easily seen that

(4) $\| y(t'; \varepsilon, p, v) - y(t''; \varepsilon, p, v) \| \leq M \, |t' - t''|$

if $\| v(t') - v(t'') \| \leq M |t' - t''|$ for t', t'' in J.

Let u_ε be an ε-polygon Euler line for (PC) on J. Evidently, $\|u_\varepsilon(t') - u_\varepsilon(t'')\| \leq M|t' - t''|$ for t', t'' in J.

Choose $\eta > 0$ and $0 < \sigma \leq 1/n$ with $\eta M + \sigma h < 1/n$. By uniform continuity of f there exists a positive $\varepsilon_0 \leq \eta$ such that $\| f(t_i, u_\varepsilon(t_i)) - f(s, u_\varepsilon(s)) \| < \sigma$ for $\varepsilon < \min(\varepsilon_0, h)$ and $t_i \leq s \leq t_{i+1}$ $(i = 1, 2, \ldots, r^0)$. Hence, by (2),

$$\| u_\varepsilon(t) - x_0 - \int_0^t f(s, u_\varepsilon(s)) ds \| \leq \int_0^{t_1} \| f(s, u_\varepsilon(s)) \| ds +$$

$$+ \sum_{m=1}^{i-1} \int_{t_m}^{t_{m+1}} \| f(t_m, u_\varepsilon(t_m)) - f(s, u_\varepsilon(s)) \| ds +$$

$$+ \int_{t_i}^{t} \| f(t_i, u_\varepsilon(t_i)) - f(s, u_\varepsilon(s)) \| ds < \eta M + \sigma h$$

for $t_i \leq t \leq t_{i+1}$. This implies

$$\sup_{t\in J}\|u_\varepsilon(t) - x_0 - \int_0^t f(s,u_\varepsilon(s))ds\| \le \eta M + \delta h < 1/n.$$

Further, by (3), for $t_j \le t' \le t_{j+1}$ and $t_k \le t'' \le t_{k+1}$ (here $j \le k$)

$$\|u_\varepsilon(t'') - u_\varepsilon(t') - \int_{t'}^{t''} f(s,u_\varepsilon(s))ds\| \le$$

$$\le \int_{t'}^{t_{j+1}} \| f(t_j,u_\varepsilon(t_j)) - f(s,u_\varepsilon(s))\| ds +$$

$$+ \sum_{m=1}^{k-j-1} \int_{t_{j+m}}^{t_{j+m+1}} \| f(t_{j+m},u_\varepsilon(t_{j+m})) -$$

$$- f(s,u_\varepsilon(s))\| ds +$$

$$+ \int_{t_k}^{t''} \| f(t_k,u_\varepsilon(t_k)) - f(s,u_\varepsilon(s)\| ds <$$

$$< \delta\,|t' - t''| \le n^{-1}|t' - t''|.$$

So we have proved the following:

Fix an index n. There exists $\varepsilon_0 > 0$ such that the Euler's ε-polygonals line $u_\varepsilon \in S_n$ for any $\varepsilon < \min(\varepsilon_0, h)$. We claim that for each $w \in S_n$ there exists a positive $\varepsilon'_0 \le \varepsilon_0$ such that (ε, p, w)-polygon Euler line $y(\cdot\,; \varepsilon,p,w) \in S_n$ for any $\varepsilon < \varepsilon'_0$ and $0 \le p \le h$.

In fact, let us assume that $\eta < \varepsilon_0$ and $\delta \le 1/n$ are such that

$$\sup_{t\in J}\| w(t) - x_0 - \int_0^t f(s,w(s))ds\| + \eta M + \delta h < 1/n.$$

By (4) we obtain $\|y(t'; \varepsilon,p,w) - y(t''; \varepsilon,p,w)\| \le M|t' - t''|$ on J. Now, similarly as above, there is a positive $\varepsilon'_0 \le \eta$ with $\|f(t_i,y(t_i; \varepsilon,p,w)) - f(s,y(s; \varepsilon,p,w))\| < \delta$ for $\varepsilon < \varepsilon'_0$, $0 \le p \le h$ and $t_i \le s \le t_{i+1}$, where $t_i = i\varepsilon$, $i = r_0 + 1,\dots,r^0 - 1$. Furthermore, let us put

$$I(t) = \| y(t; \varepsilon,p,w) - x_0 - \int_0^t f(s,y(s; \varepsilon,p,w))\, ds \|$$

for t in J. We have:

1) if $0 \le t \le p$, then $I(t) = \| w(t) - x_0 - \int_0^t f(s,w(s))ds \|$;

2) if $p \leq t \leq t_{r_o+1}$, then

$$I(t) \leq \| w(p) - x_o - \int_0^{p} f(s,w(s)) \| + $$
$$+ \| \int_{p}^{t} f(s,w(p))ds \| \leq$$
$$\leq \sup_{t\in J} \| w(t) - x_o - \int_0^t f(s,w(s))ds \| + \varepsilon M;$$

3) if $t_i \leq t \leq t_{i+1}$, then

$$I(t) \leq \sup_{t\in J} \| w(t) - x_o - \int_0^t f(s,w(s))ds \| + \varepsilon M + I_o <$$
$$< \sup_{t\in J} \| w(t) - x_o - \int_0^t f(s,w(s))ds \| + \varepsilon M +$$
$$+ \delta \sum_{m=1}^{i-r_o-1} (t_{r_o+m+1} - t_{r_o+m}) + \delta(t - t_i) \leq$$
$$\leq \sup_{t\in J} \| w(t) - x_o - \int_0^t f(s,w(s))ds \| + \varepsilon M + \delta h.$$

From this we deduce that

$$\sup_{t\in J} I(t) \leq \sup_{t\in J} \| w(t) - x_o - \int_0^t f(s,w(s))ds \| +$$
$$+ \varepsilon M + \delta h < 1/n.$$

Moreover (see (3)), for $t_j \leq t' \leq t_{j+1}$, and $t_k \leq t'' \leq t_{k+1}$,

$$\| y(t''; \varepsilon,p,w) - y(t'; \varepsilon,p,w) - \int_{t'}^{t''} f(s,y(s; \varepsilon,p,w))ds \| <$$
$$< \delta (| t_{j+1} - t' | + \sum_{m=1}^{k-j-1} | t_{j+m+1} - t_{j+m} | +$$
$$+ | t'' - t_k |) = n^{-1} | t' - t'' |.$$

Consequently

$$\| y(t''; \varepsilon,p,w) - y(t'; \varepsilon,p,w) - \int_{t'}^{t''} f(s,y(s; \varepsilon,p,w))ds \| <$$
$$< n^{-1} | t'' - t' |$$

for $t', t'' \in J$, which ends the proof.

Now, modifying the proof from [10], we prove that $\varepsilon \longmapsto u_\varepsilon$, $p \longmapsto y(\cdot\,; \varepsilon,p,w)$ (here $\varepsilon < \varepsilon'_o$) are continuous mappings of $(0, \varepsilon'_o)$ and respectively $[0,h]$ into $C(J,E)$.

For a convenience of the reader we give a short proof of the first in these results: Assume $\varepsilon(j) \to \varepsilon$ as $j \to \infty$. Let $0 < \varepsilon'$, $\varepsilon < \varepsilon'_0$ and let $t_j \le t \le t_{j+1}$, $t_i \le t \le t_{i+1}$, where $t_j = j\varepsilon'$, $t_i = i\varepsilon$ for $j = 1,2,\dots,h/\varepsilon'$ and $i = 1,2,\dots$ $\dots,h/\varepsilon$. Then

$$\begin{aligned}\|u_{\varepsilon'}(t) - u_\varepsilon(t)\| &\le \|u_{\varepsilon'}(t_j) - u_\varepsilon(t_i)\| + \\ &+ \|(t - t_j)f(t_j,u_{\varepsilon'}(t_j)) - (t - t_i)f(t_i,u_\varepsilon(t_i))\| \le \\ &\le \|u_{\varepsilon'}(t_j) - u_\varepsilon(t_i)\| + \|(t - t_j)f(t_j,u_{\varepsilon'}(t_j)) - \\ &- (t - t_i)f(t_j,u_{\varepsilon'}(t_j))\| + \\ &+ |t - t_i| \; \|f(t_j,u_{\varepsilon'}(t_j)) - f(t_i,u_\varepsilon(t_i))\| \le \\ &\le \|u_{\varepsilon'}(t_j) - u_\varepsilon(t_i)\| + M|t_j - t_i| + \\ &+ |t - t_i| \; \|f(t_j,u_{\varepsilon'}(t_j)) - f(t_i,u_\varepsilon(t_i))\|.\end{aligned}$$

This with the uniform continuity of f implies $\lim_{j\to\infty} \|u_{\varepsilon(j)}(t) - u_\varepsilon(t)\| = 0$ for each t in J, which proves that $|||u_{\varepsilon(j)} - u_\varepsilon||| \to 0$ as $j \to \infty$.

Finally, we set

$U = \{u_\varepsilon : 0 < \varepsilon < \varepsilon'_0\}$,

$V_w = \{y(\cdot\,;\varepsilon,p,w) : 0 \le p \le h\}$,

where $\varepsilon < \varepsilon'_0$ and $w \in S_n$. Note that the sets U, V_w are connected in $C(J,E)$. Furthermore, $y(\cdot\,;\varepsilon,0,w) = u_\varepsilon(\cdot) \in U \cap V_w$, $w(\cdot) = y(\cdot\,;\varepsilon,h,w) \in V_w$, and $V \subset S_n$ and $V_w \subset S_n$. The set $U \cup V_w$ is connected, and therefore the set $W_n = \bigcup\{U \cup V_w : w \in S_n\}$ is connected in $C(J,E)$. Since $S_n \subset W_n$, so $S_n = W_n$. Consequently we make the result (cf. [10]):

The set S_n $(n = 1,2,\dots)$ is nonempty and connected in $C(J,E)$.

4. <u>Main result</u>. We begin with the following two lemmas

that are of a general nature.

Lemma 1. Suppose that $u_n \in \overline{S_n}$ $(n = 1,2,\dots)$ and $(u_{k(n)})$ is a convergent subsequence of (u_n) with limit u_o. Then $u_o \in S$.

Proof. We have

$$(5) \qquad \| u_n(t) - x_o - \int_0^t f(s,u_n(s))ds \| \leq 1/n$$

$(n = 1,2,\dots)$ for t in J. Since f is uniformly continuous and $||| u_{k(n)} - u_o ||| \longrightarrow 0$ as $n \longrightarrow \infty$ it follows that $f(t,u_{k(n)}(t)) \longrightarrow f(t,u_o(t))$ uniformly on J as $n \longrightarrow \infty$. Replacing n by k(n) in (5) and letting $n \longrightarrow \infty$, we obtain $u_o(t) = x_o + \int_0^t f(s,u_o(s))ds$ for $t \in J$. It is clear from this that u_o is a solution of $x' = f(t,x)$ on J such that $u_o(0) = x_o$, which completes the proof.

Lemma 2. Let $\{X_n : n = 1,2,\dots\}$ be a family of nonempty closed and connected subsets of C(J,E) such that each sequence (x_n) with $x_n \in X_n$ contains a convergent subsequence with limit in $\bigcap_{n=1}^{\infty} X_n$. Then the set $\bigcap_{n=1}^{\infty} X_n$ is connected.

The proof follows directly from the definitions and assumptions.

We now state the main result.

Theorem. Let the function f satisfy the condition (s). Then the set S of all solutions of (PC) on J is nonempty, compact and connected in C(J,E).

Proof. By the facts above, $S = \bigcap_{n=1}^{\infty} \overline{S_n}$ and S_n $(n = 1,2,\dots)$ are nonempty connected subsets of C(J,E). Since $S \subset S_n$, so S is a compact. Let $u_n \in \overline{S_n}$ and let $(u_{k(n)})$ be a convergent subsequence of (u_n) with limit u_o. We have by Lemma 1 that $u_o \in S$. Now it follows immediately from Lemma 2 that $\bigcap_{n=1}^{\infty} \overline{S_n}$ is nonempty and connected, and the proof is finished.

5. Application. The measure of noncompactness $\alpha(X)$ of a nonempty bounded subset X of E, introduced by K. Kuratowski, is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leq \varepsilon$. (For convenience, we shall be using below the same symbol α to denote the measure of noncompactness in E as well as in other Banach spaces like C(J,E).)

Let us list some known properties of α (see e.g. [2] or [4]) which we shall use in our discussion:

Let $x \in E$ and let $A = \{p_n : n = 1,2,\dots\}$, $B = \{q_n : n = 1,2,\dots\}$ be bounded subsets of E, and let $\mathcal{X}$ be a countable bounded equicontinuous family of C(J,E). Then

1^o $\alpha(\{x\}) = 0$;

2^o if $\alpha(A) = 0$ then $\bar{A}$ is compact;

3^o $\alpha(\{tx : x \in A\}) = |t|\, \alpha(A)$ for each real t;

4^o $\alpha(\{x\} \cup A) = \alpha(A)$;

5^o $\alpha(A) - \alpha(B) \leq \alpha(\{p_n - q_n : n = 1,2,\dots\})$;

6^o if $\sup\{\|x\| : x \in A\} \leq b$, then $\alpha(A) \leq 2b$;

7^o $\sup_{t \in J} \alpha(\{y(t) : y \in \mathcal{X}\}) = \alpha(\mathcal{X})$.

From the Theorem we obtain the following result:

Let $L : J \times [0,\infty) \rightarrow [0,\infty)$ be a continuous function such that $L(0,0) = 0$ and $u(t) \equiv 0$ is the unique continuous solution of the inequality $u(t) \leq \int_0^t L(s,u(s))ds$ for which $\lim_{t \to 0+} u(t)/t$ exists and is equal to 0. Suppose that $\alpha(\{f(t,x) : x \in X\}) \leq L(t, \alpha(X))$ for any subset X of B and all t in J. Then our function f satisfies the condition (s) and consequently the set S is nonempty, compact and connected in $C(J,E)$.

<u>Proof.</u> Let $u_n \in \overline{S_n}$ for $n \geq 1$. Put $p(t) = \alpha(\{u_n(t) : n \geq 1\})$ for each t in J.

Let $t \in J$ and $t' > 0$. By 5^o and 6^o

$p(t + t') - p(t) \leq \alpha(\{u_n(t + t') - u_n(t) : n \geq 1\}) \leq 2Mt'$.

Therefore p is continuous and thus $t \mapsto L(t, p(t))$ is integrable on J. Now we prove that

$$p(t) \leq \int_0^t L(s, p(s))ds \tag{6}$$

for all t in J.

For proving (6), let $t \in J$. Since f is uniformly continuous, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that $|s - s'| < \delta$, $\| x - x' \| < \delta$ implies $\| f(s,x) - f(s',x') \| < \varepsilon/4$. For a positive integer $k > \delta^{-1} \cdot t \cdot \max(1,M)$, let $h_o = t/k$ and $s_o < s_1 < \ldots \ldots < s_k = t$ where $s_o = 0$ and $s_i = s_{i-1} + h_o$ with $i = 1,2,\ldots,k$. Then $\| f(s,u_n(s)) - f(s_i,u_n(s_i)) \| < \varepsilon/4$ $(n = 1,2,\ldots)$ for $s_{i-1} \leq s \leq s_i$ and therefore

$$\begin{aligned} &\| u_n(s_i) - u_n(s_{i-1}) - h_o f(s_i,u_n(s_i)) \| \leq \\ &\leq \| u_n(s_i) - u_n(s_{i-1}) - \int_{s_{i-1}}^{s_i} f(s,u_n(s))ds \| + \\ &+ \| \int_{s_{i-1}}^{s_i} f(s,u_n(s))ds - \int_{s_{i-1}}^{s_i} f(s_i,u_n(s_i))ds \| \leq \\ &\leq n^{-1}|s_i - s_{i-1}| + \int_{s_{i-1}}^{s_i} \| f(s,u_n(s)) - \\ &- f(s_i,u_n(s_i)) \| ds < (1/n + \varepsilon/4)h_o \leq h_o \varepsilon/2 \end{aligned}$$

for all $n \geq n_o$. Now, by $3^o - 6^o$,

$$\begin{aligned} &\sum_{i=1}^{k} (p(s_i) - p(s_{i-1}) - h_o \alpha(\{f(s_i,u_n(s_i)) : n \geq 1\})) \leq \\ &\leq \sum_{i=1}^{k} \alpha(\{u_n(s_i)) - u_n(s_{i-1}) - h_o f(s_i,u_n(s_i)) : n \geq 1\}) = \end{aligned}$$

$$= \sum_{i=1}^{k} \alpha(\{u_n(s_i) - u_n(s_{i-1})) - h_o f(s_i,u_n(s_i)) : n \geq n_o\}) \leq$$

$$\leq \sum_{i=1}^{k} 2h_o \varepsilon/2 = \varepsilon k h_o = \varepsilon t$$

and

$$- \sum_{i=1}^{k} (p(s_i) - p(s_{i-1}) - h_o \alpha(\{f(s_i,u_n(s_i)) : n \geq 1\})) \geq - \varepsilon t.$$

Hence

$$\sum_{i=1}^{k} h_o L(s_i,p(s_i)) \geq \sum_{i=1}^{k} h_o \alpha(\{f(s_i,u_n(s_i)) : n \geq 1\}) =$$

$$= \sum_{i=1}^{k} (p(s_i) - p(s_{i-1})) - \sum_{i=1}^{k} (p(s_i) - p(s_{i-1}) -$$

$$- h_o \alpha(\{f(s_i,u_n(s_i)) : n \geq 1\})) \geq (p(t_k) -$$

$$- p(t_o)) - \varepsilon t = p(t) - p(0) - \varepsilon t = p(t) - \varepsilon t.$$

Consequently

$$\int_0^t L(s,p(s))ds = \lim_{k \to \infty} \sum_{i=1}^{k} h_o L(s_i,p(s_i)) \geq p(t) - \varepsilon t.$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_0^t L(s,p(s))ds \geq p(t)$ for t in J.

It is easy to verify that $\lim_{t \to 0+} p(t)/t = 0$. By (6) and the continuity of p from this it follows that $p(t) \equiv 0$ on J. Finally $\alpha(\{u_n : n \geq 1\}) = \sup_{t \in J} \alpha(\{u_n(t) : n \geq 1\}) = 0$, since $\{u_n : n \geq 1\}$ is a bounded equicontinuous family. Hence $\{u_n : n \geq 1\}$ is conditionally compact, and we are done.

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