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# A LIOUVILLE THEOREM FOR NONLINEAR ELLIPTIC SYSTEMS WITH ISOTROPIC NONLINEARITIES

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Abstract: We show that if $u = (u_1, \dots, u_m)$ is a solution with bounded gradient in $\mathbb{R}^n$ of an elliptic system of the form:

$$- \frac{\partial}{\partial x_i} \left( a_{ij}(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_j} \right) = 0, \; 1 \le \alpha \le m,$$

then each $u_\alpha$ is an affine function on $\mathbb{R}^n$.

Key words: elliptic systems, Liouville theorem, regularity, Harnack inequality.

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## I Introduction:

We consider here a nonlinear second-order elliptic system of the following form:

$$(1) \qquad - \frac{\partial}{\partial x_i} \left( a_{ij}(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_j} \right) = 0 \text{ in } \mathbb{R}^n, \; u = (u_1, \dots, u_m), \; 1 \le \alpha \le m.$$

Throughout all the paper we will assume that $a_{ij} \in C^1(\mathbb{R})$ (for $1 \le i,j \le n$) and that (1) is very strongly elliptic in the sense that for every $\eta$ and $\xi \neq 0$

$$(2) \qquad a_{ij}(|\eta|^2)\, \xi_i^\alpha \xi_j^\alpha + 2\, a'_{ik}(|\eta|^2)\, \eta_i^\alpha \eta_j^\beta \xi_i^\alpha \xi_j^\beta > 0$$

We prove below that if u has a bounded gradient on $\mathbb{R}^n$, then each component $u_\alpha$ of $u$ is affine on $\mathbb{R}^n$.

This result is clearly a Liouville type theorem. Let us explain now how this result is related to various facts from nonlinear second-order elliptic systems theory. To this end, let us consider a general second order elliptic system:

$$(3) \qquad -\frac{\partial}{\partial x_i}\left(a_i^\alpha(x,u,\nabla u)\right) + a^\alpha(x,u,\nabla u) = f^\alpha(x) \text{ in } \Omega$$

where $1 \leq \alpha \leq m$, $u = (u_1,\dots,u_m)$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$. The very strong ellipticity of the system (3) is expressed by the following condition:

$$(4) \qquad \frac{\partial a_i^\alpha}{\partial \gamma_j^\beta}(x,\xi,\gamma)\, \xi_i^\alpha \xi_j^\beta > 0, \qquad \xi \neq 0.$$

Of course, when (3) reduces to (1), (4) is nothing else than (2). Assuming that $u$ is a Lipschitz solution of (3), one may ask the following natural (and fundamental) question: is $u$ of class $C^1$ or even $C^{1,\mu}$ (for some $\mu \in (0,1)$) ?

As shown by M.Giaquinta and J.Nečas [2], this regularity question turns out to be, in some sense, equivalent to the following Liouville type condition: (3) is said to satisfy the Liouville condition (in short $L(\mathbb{R}^n)$) provided the following implication holds: for all $x^0 \in \Omega$, $\xi \in \mathbb{R}^m$, if $v = (v_1,\dots,v_m)$ is a solution with bounded gradient of

$$(3') \qquad -\frac{\partial}{\partial x_i}\left(a_i^\alpha(x^0,\xi,\nabla v)\right) = 0 \quad \text{in } \mathbb{R}^n,$$

then each $v$ is affine on $\mathbb{R}^n$. More precisely, in [2] it is proved that if the system (3) (where we assume (4) with $a_i^\alpha$, $a^\alpha \in C^1(\bar{\Omega} \times R^m \times \mathbb{R}^{mn})$ satisfies $L(\mathbb{R}^n)$ and $p > n$, then for every $\nu > 0$ and every compact set $K \subset \Omega$ there is $c(\nu,K) < \infty$ such that

(5) $$\|u_\alpha\|_{C^{1,\mu}(K)} \leq c(\nu, K), \qquad 1 \leq \alpha \leq m,$$

with $\mu = 1 - (n/p)$, whenever $f^\alpha \in L^p(\Omega)$ and $u$ is a Lipschitz solution of (3) such that

$$\|u\|_{[W^{1,\infty}(\Omega)]^m} + \|f\|_{[L^p(\Omega)]^m} \leq \nu .$$

Conversely, in some sense, $L(\mathbb{R}^n)$ is a consequence of regularity results of the form (5) - see J.Nečas [6],[7] or M.Giaquinta [1].

Therefore the Liouville result we prove in this paper immediately yields the $C^{1,\mu}$ regularity for special systems of form:

(6) $$-\frac{\partial}{\partial x_i}\left(a_{ij}(x,u,|\nabla u|^2)\frac{\partial u_\alpha}{\partial x_j}\right) + a^\alpha(x,u,\nabla u) = f^\alpha(x) \text{ in } \Omega$$

**(for $1 \leq \alpha \leq m$). At this point, we want to point** out that this regularity result (a consequence of our result and an equivalent when $a_{ij}$ depend on $|\nabla u|^2$ only) was established by P.A.Ivert [4] in a generalization of deep results due to K.Uhlenbeck [8]. Thus, in some sense, the result we present here is not new and could be derived from Uhlenbeck - Ivert results. On the other hand, our method of proof is quite different from those of [4], [8] and, we believe, much simpler. Let us also mention that it is straightforward to adapt our method of proof to show directly the $C^{1,\mu}$ regularity result (looking, roughly speaking, at little balls instead of large balls).

Let us conclude this introduction by a few words on our method of proof. In section II below, we present a general result on nonlinear elliptic systems which implies in particular that, if we denote by $\omega = \nabla u$, we have: there is $\varepsilon_0 > 0$ such that if $\Phi_\omega(R) < \varepsilon_0$, then for every $\rho \in (0,R)$

(7) $$\Phi_\omega(\rho) \leq C_0\, \Phi_\omega(R)$$

where $C_0$ depends only on $\|\omega\|_{L^\infty(B_R)}$ and where for a vector valued function $g$ we denote:

$$\Phi_g(\rho) = \frac{1}{\rho^n} \int_{B_\rho} |g(x) - (g)^\rho|^2 \, dx,$$

$$(g)^\rho = (1/|B_\rho|) \int_{B_\rho} g(x)dx.$$

By an easy use of Poincaré inequality, we see that in order to conclude (using(7)) we just need to show that $\omega = \nabla u$ has the so-called Saint-Venant property:

$$\lim_{R \to \infty} R^{-n+2} \int_{B_R} |\nabla \omega(x)|^2 \, dx = 0 \quad . \tag{8}$$

The main idea used to prove (7) goes back to a fundamental lemma of E.Giusti - see e.g. [2] .

Next, in section III, we state and prove a Liouville type theorem. This is done by remarking - following [4],[8] - that $|\nabla u|^2 = w$ satisfies:

$$- \frac{\partial}{\partial x_i} (A_{ij} \frac{\partial w}{\partial x_j}) + \alpha |D^2 u|^2 \leq 0 \quad \text{in} \quad \mathbb{R}^n \tag{9}$$

for some $\alpha > 0$, and for some uniformly elliptic coefficients $A_{ij}$. Using this inequality and a Harnack type inequality proved in D.Gilbarg and N.S.Trudinger [3] (for example), we show that (8) holds and thus $\omega$ is constant.

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## II A general result on quasilinear elliptic systems:

In this section we consider a solution $\omega = (\omega_1, \dots, \omega_N)$ of

(10) $$-\frac{\partial}{\partial x_i}\left[A_{ij}^{\alpha\beta}(\omega)\frac{\partial \omega_\beta}{\partial x_j}\right] = 0 \quad \text{in } \mathbb{R}^n, \quad \alpha = 1,\dots,N,$$

where $A_{ij}^{\alpha\beta}$ are continuous on $\mathbb{R}^m$ and where the ellipticity condition

(11) $$A_{ij}^{\alpha\beta}(\xi)\,\xi_i^\alpha \xi_j^\beta > 0 \quad \text{for } \xi \neq 0$$

holds.

Theorem II. 1: Let $R > 0$, let $\omega$ be a bounded solution of (10) in $(H^1(B_R))^N$ and let us assume that (11) holds. We denote $\mu = \|\omega\|_{L^\infty(B_R)}$. Then there exist $\varepsilon_0 > 0$, $C_0 > 0$ such that the following statement holds:

if $$\Phi_\omega(R) \leq \varepsilon_0^2,$$

then $$\Phi_\omega(\rho) \leq C_0\,\Phi_\omega(R)$$

whenever $\rho \in (0,R)$. In addition $\varepsilon_0$, $C_0$ depend only on $\mu$ and on the ellipticity constants in (11).

Before giving the proof of Theorem II.1, let us mention the

Corollary II.1: Let $\omega$ be a bounded solution of (10) in $(H^1_{loc}(\mathbb{R}^n))^N$ satisfying the Saint-Venant property

$$\lim_{R\to\infty} R^{-n+2}\int_{B_R} |\nabla\omega(x)|^2\,dx = 0,$$

and let us assume that (11) holds. Then $\omega$ is a constant vector.

Proof: Observe that we have by Poincaré inequality:

(12) $$R^{-n}\int_{B_R} |\omega(x) - (\omega)^R|^2\,dx \leq c_1 R^{-n+2}\int_{B_R} |\nabla\omega(x)|^2\,dx.$$

(Here and below $c_1, c_2, \dots$ denote various positive contents independent of $R, \omega, u$.) Thus we see that (8) implies: $\lim_{R\to\infty}\Phi_\omega(R) = 0$. Therefore by Theorem II.1, $\Phi_\omega(\rho) = 0$ for all $\rho > 0$ and the proof is complete.

Proof of Theorem II.1: First of all, in view of (11), there exists $\nu > 0$ such that for every $\xi$ and $|\xi| \leq \mu$ we have

$$[A_{ij}^{\alpha\beta}(\xi)\, A_{ij}^{\alpha\beta}(\xi)]^{1/2} \leq \frac{1}{\nu}, \quad A_{ij}^{\alpha\beta}(\xi)\, \xi_i^{\alpha} \xi_j^{\beta} \geq \nu\, |\xi|^2.$$

Let us also recall that it is known (see e.g.[2]) that there exists $c_2$ ( $= c_2(\mu, \nu)$) such that we have:

$$\Phi_{\omega}(\tau) \leq c_2 \tau^2 . \Phi_{\omega}(1), \qquad 0 < \tau \leq 1, \tag{13}$$

if $\omega$ is a solution of the system:

$$-\frac{\partial}{\partial y_i}\left(A_{ij}^{\alpha\beta}(\xi)\frac{\partial \omega_\beta}{\partial y_j}\right) = 0 \quad \text{in } B_1$$

where $|\xi| \leq \mu$.

Next, let $\tau \in (0,1)$. We are first going to prove that there exist $\varepsilon_0 = \varepsilon_0(\mu, \tau, \nu) > 0$ such that

$$\Phi_{\omega}(\tau) \leq 2\, c_2 \tau^2\, \Phi_{\omega}(1) \tag{14}$$

where $\omega$ solves (10) and satisfies: $\|\omega\|_{L^\infty(B_1)} \leq \mu$, $\Phi_\omega(1) \leq \varepsilon_0^2$.

Let us argue by contradiction and let us thus assume that there exists a sequence $(\omega^n)_{n\geq 1}$ of solutions of (10) satisfying:

$$\|\omega^n\|_{L^\infty(B_1)} \leq \mu, \ \{\Phi_{\omega^n}(1)\}^{1/2} = \varepsilon_n \longrightarrow 0, \tag{15}$$

$$\Phi_{\omega^n}(\tau) > 2c_2 \tau^2 \varepsilon_n^2.$$

To simplify notations, we will use indifferently the notations $\Phi_{\omega^n}(\tau)$ or $\Phi(\omega^n, \tau)$. We then set: $\sigma^n = \frac{1}{\varepsilon_n}[\omega^n - (\omega^n)_1]$.

Obviously we have:

$$\int_{B_1} |\sigma^n(x)|^2 dx = 1; \ \Phi(\sigma^n, \tau) > 2c_2 \tau^2; \tag{16}$$

(17) $$- \frac{\partial}{\partial x_i} \left( A_{ij}^{\alpha\beta}(\omega^n) \frac{\partial \sigma_\beta^n}{\partial x_j} \right) = 0 .$$

Without loss of generality we may assume that:

$$\sigma^n \longrightarrow \sigma \text{ weakly in } (L^2(B_1))^N, \quad \varepsilon_n \sigma^n \longrightarrow 0 \quad \text{in} (L^2(B_1))^N \text{ and a.e.,}$$

for some $\sigma \in (L^2(B_1))^N$. In addition, in wiew of (16): $\Phi_\sigma(1) \leq 1$. Furthermore, recalling that we have:

$$\omega^n = \varepsilon_n \sigma^n + (\omega^n)^1, \ \| \omega^n \|_{L^\infty(B_1)} \leq \mu ,$$

we see that $|(\omega^n)^1| \leq \mu$ and $\omega^n - (\omega^n)^1 \longrightarrow 0$ a.e. Since we may assume without loss of generality that $(\omega^n)^1 \longrightarrow \xi$ ($|\xi| \leq \mu$), we finally deduce: $\omega^n \longrightarrow \xi$ . a.e. .

Next, we obtain from (16) and (17):

(18) $$\int_{B_k} |\nabla \sigma^n(y)|^2 \, dy \leq C(k) \quad \text{for} \quad k \in (0,1),$$

thus we may suppose that $\sigma^n \longrightarrow \sigma$ weakly in $(H^1(B_k))^N$ (for all $k < 1$). Thus, passing to the limit in (17), we get:

$$- \frac{\partial}{\partial x_i} \left( A_{ij}^{\alpha\beta}(\xi) \frac{\partial \sigma_\beta}{\partial x_j} \right) = 0 \quad \text{in} \quad B_1 .$$

In addition, since $\sigma_n \longrightarrow \sigma$ in $(L^2(B_k))^N$ (for all $k < 1$), we deduce from (16): $\Phi(\sigma, \tau) \geq 2 c_2 \tau^2 \geq 2c_2 \tau^2 \Phi(\sigma, 1)$. This contradicts (13) and the contradiction shows our claim.

Let us choose now $\tau \in (0,1)$ satisfying: $2c_2 \tau^2 \leq 1$. Given $\rho \in (0,1)$, let $k \geq 0$ be the integer such that: $\tau^{k+1} \leq \rho < \tau^k$. Now, if $\omega$ solves (10) and satisfies: $\| \omega \|_{L^\infty(B_1)} \leq \mu$, $\Phi_\omega(1) \leq \varepsilon_0^2$, we have in view of (14):

$$\tau^n \rho^{-n} \int_{B_\rho} |\omega - (\omega)^\rho|^2 \, dx \leq (\rho / \tau^k)^n \rho^{-n} \int_{B_\rho} |\omega - (\omega)^\rho|^2 \, dx \leq$$

$$\leq (\tau^k)^{-n} \int_{B_\rho} |\omega - (\omega)^{\tau^k}|^2 \, dx \leq (\tau^k)^{-n} \int_{B_{\tau^k}} |\omega - (\omega)^{\tau^k}|^2 \, dx \leq$$

$$\leq \int_{B_1} |\omega - (\omega)^1|^2 \, dx;$$

that is, we proved: $\Phi_\omega(\rho) \leq \tau^{-n} \Phi_\omega(1)$.

The proof of Theorem II.1 is easily completed by considering the function $\tilde{\omega}(x) = \omega(x/R)$.

Remark II.1: We now show how the preceding results are related to the system (1): indeed, if $u \in (H^2_{loc}(\mathbb{R}^n))^m$ is a solution of (1) then, for $1 \leq k \leq n$, $\frac{\partial u_\alpha}{\partial x_k}$ satisfies:

$$- \frac{\partial}{\partial x_i} \left[ A^{\alpha\beta}_{ij}(\nabla u) \frac{\partial}{\partial x_j} \left( \frac{\partial u_\beta}{\partial x_k} \right) \right] = 0 \text{ in } \mathbb{R}^n, \quad 1 \leq \alpha \leq m,$$

where $A^{\alpha\beta}_{ij}(\nabla u) = a_{ij}(|\nabla u|^2) \delta_{\alpha\beta} + 2\, a'(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_\ell} \frac{\partial u_\beta}{\partial x_j}$ .

Thus $\omega = \nabla u$ satisfies a system of the form (10) and (11) is a consequence of (2).

## III The main result:

Let $u = (u_1, \dots, u_m)$ be a solution of (1):

$$- \frac{\partial}{\partial x_i} \left( a_{ij}(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_j} \right) = 0 \text{ in } \mathbb{R}^n, \quad 1 \leq \alpha \leq m.$$

Theorem III.1: We assume the ellipticity condition (2) and $\nabla u \in (L^\infty(\mathbb{R}^n))^{nm}$. Then each component $u_\alpha$ of $u$ is affine on $\mathbb{R}^n$.

Proof: Standard arguments yield $u \in W^{2,2}_{loc}(\mathbb{R}^n)$; cf. [7] or [1] . In view of the results of the preceding section and of Remark II.1, it is enough to show:

(19) $$\lim_{R \to \infty} R^{-n+2} \int_{B_R} |D^2 u|^2 \, dx = 0.$$

In order to prove (19), we first observe that an easy computation yields:

$$- \frac{\partial}{\partial x_i} \left[ A_{ij}(\nabla u) \frac{\partial}{\partial x_j}(|\nabla u|^2) \right] + a_{ij} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_k} \frac{\partial^2 u_\alpha}{\partial x_j \partial x_k} +$$

(20)

$$+ 2 a'_{ik} \frac{\partial u_\alpha}{\partial x_k} \frac{\partial u_\beta}{\partial x_j} \frac{\partial^2 u_\alpha}{\partial x_i \partial x_s} \frac{\partial^2 u_\beta}{\partial x_j \partial x_s} = 0 \quad ,$$

where $A_{ij}(\nabla u) = \frac{1}{2} a_{ij}(|\nabla u|^2) + a'_{ik}(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_k} \frac{\partial u_\alpha}{\partial x_j}$

In view of (2), we see that (for more details, see [4])

(21) $\exists \nu > 0, \ \forall \xi \in \mathbb{R}^n, \ A_{ij}(\nabla u(x)) \xi_i \xi_j \geq \nu |\xi|^2$ ,

$$\{A_{ij}(\nabla u(x)) A_{ij}(\nabla u(x))\}^{1/2} \leq \frac{1}{\nu} \text{ a.e. in } \mathbb{R}^n$$

and (20) implies:

(22) $$- \frac{\partial}{\partial x_i} (A_{ij}(\nabla u) \frac{\partial}{\partial x_j} (|\nabla u|^2)) + \alpha |D^2 u|^2 \leq 0 \text{ in } \mathbb{R}^n,$$

for some $\alpha > 0$. We denote $M = \| |\nabla u|^2 \|_{L^\infty(\mathbb{R}^n)}$.

We are now going to prove:

(23) $$R^{-n+2} \int_{B_{R/2}} |D^2 u|^2 \, dx \leq c_3 R^{-n} \int_{B_{2R}} (M - |\nabla u|^2) dx.$$

To this end we introduce $\eta \in H_0^1(B_{2R})$, the solution of:

(24) $$- \frac{\partial}{\partial x_i} (A_{ji} \frac{\partial \eta}{\partial x_j}) = \frac{1}{R^2} \text{ in } B_{2R} \quad .$$

Standard results yield: $\eta \geq 0$ in $B_{2R}$ and

(25) $\|\eta\|_{L^\infty(B_{2R})} \leq c$ $\qquad \inf_{B_{R/2}} \eta \geq c_5 > 0.$

Then multiplying (22) by $\psi^2$ and using (24), (25), we deduce:

$$c_6 \int_{B_{R/2}} |D^2u(x)|^2 \, dx \leq \int_{B_{2R}} A_{ij} \frac{\partial \psi^2}{\partial x_i} \frac{\partial}{\partial x_j} (M - |\nabla u|^2) \, dx \leq$$

$$\leq 2 \int_{B_{2R}} \frac{1}{R^2} \psi (M - |\nabla u|^2) \, dx - 2 \int_{B_{2R}} A_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} (M - |\nabla u|^2) dx$$

and this yields:

$$\int_{B_{R/2}} |D^2u(x)|^2 \, dx \leq \frac{c_7}{R^2} \int_{B_{2R}} (M - |\nabla u|^2) dx$$

and (23) is proved.

To conclude, we see that (19) follows from (23), applying the following lemma to $w = |\nabla u|^2$, $\alpha_{ij}(x) = A_{ij}(\nabla u(x))$.

Lemma III.1: Let $w \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy:

$$- \frac{\partial}{\partial x_i} \left( \alpha_{ij}(x) \frac{\partial w}{\partial x_j} \right) \leq 0 \quad \text{in } \mathbb{R}^n$$

where $\alpha_{ij} \in L^\infty(\mathbb{R}^n)$ satisfy:

$$\{\alpha_{ij}(x) \, \alpha_{ij}(x)\}^{1/2} \leq \frac{1}{\nu}, \quad \alpha_{ij}(x) \, \xi_i \, \xi_j \geq \nu |\xi|^2 \quad \forall \xi \in \mathbb{R}^n,$$

a.e. in $\mathbb{R}^n$

for some $\nu > 0$. If $M = \sup\text{ess}_{\mathbb{R}^n} \, w$, then we have:

(26) $$\lim_{R \to \infty} (1/|B_R|) \int_{B_R} w(x) dx = M.$$

Proof: This lemma is proved by the use of a weak Harnack inequality (cf.[3], for example) which implies:

(27) $$R^{-n} \int_{B_{2R}} z(x) \, dx \leq c_8 \inf\text{ess}_{B_R} \; z$$

with $z = M - w$. Now if we let $R \to \infty$, we obtain (26) since

$\inf\limits_{B_R} \text{ess}\ z \longrightarrow \inf\limits_{\mathbb{R}^n} \text{ess}\ z = 0$; and $z \geq 0$ a.e.in $R^n$.

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