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RESTRICTED MEAN VALUE PROPERTY IN AXIOMATIC POTENTIAL THEORY

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Abstract: A generalization of the restricted mean value property and a corresponding converse of the Gauss mean value theorem are studied in the frame of the axiomatic potential theory.

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Introduction. Let $G \subset R^n$ be an open set having non-empty boundary ∂G and let δ be an arbitrary function on G such that $0 < \delta(x) < \mathrm{dist}(x, \partial G)$, the distance of x from the boundary. A Borel measurable function g on G is said to have the restricted mean value property with respect to balls, if for every $x \in G$ the mean value of g over the ball with centre x and radius $\delta(x)$ (i.e. the integral of g over the ball divided by its volume) is equal to g(x). A function g with the described property is sometimes called δ-harmonic function on G. Not so frequent is the study of δ-harmonicity with respect to spheres where the mean value may be interpreted as the integral with respect to harmonic measure, nevertheless this serves to us as a starting point for our generalization of the problem.

A quite natural generalization in the context of a harmonic space X may be formulated as follows: For every x from an open set $G \subset X$ fix a measure ν_x such that for every function $h \in \mathcal{H}(G)$, the set of harmonic functions on G, we have

$$\nu_x(h) = \int h d\nu_x = h(x).$$

Now the problem consists of finding conditions involving G, g and the choice of measures ν_x which imply $g \in \mathcal{H}(G)$ provided $\nu_x(g) = g(x)$ for every $x \in G$. A lot of results is known for the classical case and various conditions imposed on G, g and σ which give the validity of

$$g \text{ is } \sigma\text{-harmonic} \Longrightarrow g \in \mathcal{H}(G)$$

were studied (see, e.g. [5],[12],[4]; a survey article [9] contains 81 references concerning the problem). As far as we know in the frame of an axiomatics such a problem has not been studied. Our approach to it is closely connected with the Dirichlet problem and avoids probabilistic methods which occurred to be very efficient in the classical case. Its origin goes back to Lebesgue [7] (this influenced our terminology). Other comments may be found at the end of the article.

1. Notation. Let X be a $\mathfrak{P}$-harmonic space with countable base in the sense of [3]. In what follows, $G \subset X$ is a fixed open set for which $\partial G \neq \emptyset$ For a set $M \subset X$, the symbol $\mathcal{C}(M)$ stands for the set of continuous functions on M while $\mathcal{B}(M)$ denotes the set of Borel measurable functions on M. The set of functions from $\mathcal{C}(X)$ having compact support is denoted by $\mathcal{C}_c(X)$ and $\mathcal{B}_b(M)$ is the subset of $\mathcal{B}(M)$ consisting of bound-

ed functions.

For an open relatively compact $G \subset X$ let us denote by $S(\overline{G}) = S$ the cone of lower semicontinuous functions (l.s.c.) on $\overline{G}$ which are extensions of superharmonic functions on G. For every $x \in \overline{G}$ choose a representing measure α_x with respect to S, i.e. a measure α_x such that

$$(1) \qquad s \in S \Rightarrow \alpha_x(s) \leqq s(x).$$

Redefining for a function $s \in S$ the value $s(x)$ for an $x \in \partial G$ in such a way that $s(x) < s(y)$ for all $y \in \overline{G} \setminus \{x\}$ we can easily show $\alpha_x = \varepsilon_x$ (the Dirac measure); hence $\alpha_x = \varepsilon_x$ for all $x \in \partial G$.

Now we can put for $f \in \mathcal{B}_b(G)$

$$(2) \qquad Af(x) := \alpha_x(f)$$

where we identify A with the collection $\{\alpha_x\}_{x \in \overline{G}}$. Since mostly the choice of α_x together with (2) does not determine a "good operator" A, we must be a bit careful: $Af = f$ means only that f is α_x-integrable for all x and

$$(3) \qquad \alpha_x(f) = f(x).$$

If (2) determines an operator acting on the space $\mathcal{C}(\overline{G})$ or $\mathcal{B}_b(\overline{G})$, we shall call A "continuous" or "Borel" operator, respectively. Our first aim will be to find sufficient conditions for the validity of

$$(4) \qquad Af = f \Rightarrow f|G \in \mathcal{H}(G);$$

here $f|G$ means the restriction of f on G. Then we shall use it in connection with the restricted mean value property.

2. Lebesgue's operators. Let G be a relatively compact open set. For $A = \{\alpha_x\}_{x\in\overline{G}}$ suppose (1) and

(5) $$\alpha_x \neq \varepsilon_x \text{ for all } x \in G.$$

In what follows we shall need a variant of the minimum principle:

If S_A is the set of l.s.c. functions s on the compact set $\overline{G}$ for which $As \leqq s$, then for every $s \in S_A$ we have

(6) $$s \mid \partial G \geqq 0 \Rightarrow s \geq 0.$$

Suppose, on the contrary, that there is a function $s \in S_A$ such that $s(x) < 0$ for an $x \in G$ and $s \mid \partial G \geq 0$. If $p > 0$ is a continuous potential on X, put $f = s$, $g = p\mid\overline{G}$ in Lemma 2 ([3], p. 26) to obtain a contradiction with (5). By a Lebesgue's operator $A = \{\alpha_x\}_{x\in\overline{G}}$ (shortly: L-operator) we understand a continuous operator defined by (2) provided (1) and (5) hold. Denote by α_x^n the measure determined for all x and n with the help of the equality

(7) $$\alpha_x^n(f) := A^n f(x), \quad f \in \mathscr{C}(\overline{G})$$

(here A^0 is the identity operator and $A^n = A * A^{n-1}$, $n = 1,2, \ldots$, where $*$ stands for the composition of operators) and by μ_x the harmonic measure corresponding to G and x.

3. Proposition. Let $G \subset X$ be an open relatively compact set and $A = \{\alpha_x\}_{x\in\overline{G}}$ be an L-operator. If α_x^n are defined by (7), then for every $f \in \mathscr{C}(\overline{G})$

(8) $$\alpha_x^n(f) \longrightarrow \mu_x(f), \quad x \in G$$

as $n \longrightarrow \infty$ (i.e. $\alpha_x^n \longrightarrow \mu_x$ weakly) and the convergence in (8) is uniform with respect to x on compact subsets of G.

Proof. Since the differences of continuous potentials are uniformly dense in $\mathcal{C}_c(X)$ (cf. [3], Th. 2.3.1), we restrict ourselves to the case of $f \in \mathcal{C}(\overline{G})$ and $f|G$ superharmonic. Then we have

$$f_1 := Af \leqq f, \qquad f_2 := Af_1 \leqq f_1, \ldots$$

and $\{f_n\}$ is lower bounded. Put $u = \lim f_n$. It is easily seen that $v(x) := \mu_x(f) \leqq u(x)$, $x \in G$ and $Au = u$.

If s is an upper function for $f \mid \partial G$, we extend the difference $(s-u)$ by lim inf from G onto $\overline{G}$. Then $s-u \in S_A$ and by (6) we have $u \leqq s$. Consequently $u = v$. The rest follows from the Dini's theorem (cf. [13], Th. 6).

4. <u>Corollary</u>. Let $G \subset X$ be an open relatively compact set. Suppose that for a Borel operator $A = \{\alpha_x\}_{x \in \overline{G}}$ there is an L-operator $B = \{\beta_x\}_{x \in \overline{G}}$ for which

(9) $$s \in S \Longrightarrow \alpha_x(s) \leqq \beta_x(s), \qquad x \in G.$$

Then (8) is valid for every $f \in \mathcal{C}(\overline{G})$.

Indeed, it is easily seen that for every $s \in S$ we have $\mu_x(s) \leqq \alpha_x^n(s) \leqq \beta_x^n(s) \longrightarrow \mu_x(s)$ and hence the statement is an easy consequence of (9). In connection with the Dirichlet problem both previous assertions offer the solution as a limit of a sequence of functions. More precisely, if $f \in \mathcal{C}(\overline{G})$ is an extension of the boundary condition in the Dirichlet problem and A is as above, then $A^n f$ is convergent on G to the solution of the problem (see [13]).

Now we shall turn to the restricted mean value property. We shall suppose

(10) $\qquad \operatorname{spt} \alpha_x \subset G$ for all $x \in G$;

this assumption is quite natural with regard to the classical case we discussed in the introduction. The same may be said on the definition:

If $G \subset X$ is open and $A = \{\alpha_x\}_{x \in G}$ is a collection of measures having compact support and satisfying (1), (5) and (10), then a locally bounded $f \in \mathcal{B}(G)$ is said to be A-harmonic on G provided (3) holds for every $x \in G$.

5. <u>Proposition</u>. Suppose that $G \subset X$ is an open relatively compact set and $A = \{\alpha_x\}_{x \in G}$ is a Borel operator (i. e. $Ag \in \mathcal{B}_b(G)$ whenever $g \in \mathcal{B}_b(G)$) for which there is an L-operator $B = \{\beta_x\}_{x \in \overline{G}}$ such that (9) is valid. If $f \in \mathcal{B}_b(\overline{G})$ and f is μ_x - almost everywhere continuous on $\overline{G}$ for every $x \in G$, then from A-harmonicity of f follows $f|G \in \mathcal{H}_b(G)$.

This is a consequence of Cor. 4, namely the weak convergence $\alpha_x^n \longrightarrow \mu_x$, which we established there and from which it follows for f

$$f(x) = \alpha_x^n(f) = \mu_x(f), \qquad x \in G.$$

(See [1], Th. 4.5.1.)

The condition (9) shows that for every $x \in G$ the corresponding α_x is balayed β_x with respect to the cone S. Roughly speaking, A-harmonicity implies harmonicity if A is a Borel operator (on $\mathcal{B}_b(G)$) which is a "pointwise balayed L-operator".

It seems a bit inconvenient to work with an f defined on

$\bar{G}$ since the result is naturally connected with its values on G. This is removed by the next theorem.

6. <u>Theorem</u>. Let $G \subset X$ be an open relatively compact set. Suppose that $f \in \mathcal{B}_b(G)$ is an A-harmonic function on G with respect to the Borel operator $A = \{\alpha_x\}_{x \in G}$. Moreover, suppose that (i) there is an L-operator $B = \{\beta_x\}_{x \in \bar{G}}$ and (9) holds, (ii) there is a Borel set $N_f \subset \partial G$ such that

$$(11) \qquad \lim_{x \to y} f(x)$$

is finite for every $y \in \partial G \setminus N_f$ and (iii) the set N_f is negligible in the sense that

$$(12) \qquad \mu_x(N_f) = 0 \text{ for every } x \in G.$$

Then f is a bounded harmonic function on G.

Proof. Consider the weak convergence $\alpha_x^n \longrightarrow \mu_x$ in (8) for a fixed $x \in G$. Recall that by (10) $\operatorname{spt} \alpha_x^n \subset G$ while for the harmonic measure we have $\operatorname{spt} \mu_x \subset \partial G$. Put $P = \bar{G} \setminus N_f$, $\alpha_x^n | P = \alpha_n$, $\mu_x | P = \gamma$. It is easily seen that the function f extended from G on P by limits (11) is a Borel measurable function which is γ-almost everywhere continuous on P. It remains to prove $\alpha_n \longrightarrow \gamma$ weakly and again to apply Th. 4.5.1 in [1], where also equivalent conditions for the weak convergence of measures may be found. For any $V \subset P$ open there is an open $V' \subset \bar{G}$ such that $V' \cap P = V$. Since we know (8), we can easily prove $\liminf_{n \to \infty} \alpha_n(V) \geqq \gamma(V)$ and the required weak convergence will be established. Indeed, $\liminf \alpha_n(V) = \liminf \alpha_x^n(V) =$ $= \liminf \alpha_x^n(V') \geqq \mu_x(V') = \mu_x(V) = \gamma(V)$, and hence $\alpha_n \longrightarrow$ $\longrightarrow \gamma$ is proved.

Let us remark that it is not obvious whether (12) is equivalent with $\nu(N_f) = 0$ for a measure ν. But it is the case if we choose a countable dense subset $\{x_n; n \in N\} \subset G$ and put $\nu = \sum 2^{-n} \mu_{x_n}$. Since there is a continuous potential $p > 0$ on X, it is easily seen that the measures μ_{x_n} are uniformly bounded and hence ν is a finite measure. Now, if (12) holds, $\nu(N_f) = 0$ is obvious. On the other hand, if $\nu(M) = 0$ for an $M \subset \partial G$, then $\mu_{x_n}(M) = 0$ for every $n \in N$. Since $\mu_x(M)$ is - as a function of x on G - harmonic and therefore continuous on G, it is equal 0 everywhere and (12) is valid.

To find a connection of the theorem to the results known in the classical case, we shall at first examine collections A of "typical" α_x. If V_x is a neighbourhood of $x \in G$ such that $\overline{V}_x \subset G$ is compact, then a good candidate for such an α_x is $\varepsilon_x^{CV_x}$ (the harmonic measure corresponding to the set V_x at x). Since the support of the measure is compact, every $f \in \mathcal{B}_b(G)$ will be α_x-integrable. In what follows, U, V (sometimes with primes or indexes) will stand for open relatively compact subsets of G. Let us fix a metric ϱ on X compatible with its topology and introduce

$$\sigma(U,V) = \tilde{\varrho}(\overline{U},\overline{V}) + \tilde{\varrho}(\partial U, \partial V),$$

where $\tilde{\varrho}$ is the Hausdorff metric on the system of compact subsets of X. It is easily seen that σ is a metric on the set of open relatively compact subsets of G. Moreover, if $\varepsilon > 0$ is given and for a U we put

$V' = \{x \in U; \operatorname{dist}(x,CU) > \varepsilon\}$, $V'' = \{x \in G; \operatorname{dist}(x,U) < \varepsilon\}$,

then $V' \subset U \subset V''$ and for every V, $\sigma(U,V) < \varepsilon$ we have

$V' \subset V \subset V''$. For a technical reason let us put

$$\alpha(U,x,f) := \int f \, d\varepsilon_x^{CU}$$

for $x \in U$, $\overline{U} \subset G$ and $f \in \mathcal{B}_b(G)$.

7. <u>Proposition</u>. Let $G \subset X$ be an open set. Suppose that the mapping $\omega : x \longrightarrow V_x$, $x \in G$ has the following properties;

(i) V_x is a neighbourhood of x such that $\overline{V}_x \subset G$ is compact and $\varepsilon_x^{CV_x} = \varepsilon_x^{C\overline{V}_x}$,

(ii) ω is continuous (w.r.t. $\rho - \sigma$).

Then, for every $f \in \mathcal{C}(G)$, the function

$$Af(x) := \alpha(V_x,x,f), \quad x \in G$$

is continuous (here $\alpha(V_x,x,\cdot)$ denotes a measure previously denoted by α_x).

Remark. The equality of measures in (i) implies certain continuity: put $V = V_x$ and choose U_n, U_n' in such a way that $U_n \nearrow V$, $U_n' \searrow V$. Then it follows from Pr. 7.2.4 in [3] that $\varepsilon_y^{CV_n} \longrightarrow \varepsilon_y^{CV}$, $\varepsilon_y^{CV_n'} \longrightarrow \varepsilon_y^{C\overline{V}}$ for every $y \in V$ and hence our assumption implies that both limits coincide for $y = x$. (The condition is fulfilled if the set of points of ∂V in which $C\overline{V}$ is thin is "small"; cf. [10], Th.1.)

Proof. Let us prove the continuity of Af at arbitrarily chosen $x \in G$. For such a fixed x choose U, $\overline{V}_x \subset U \subset \overline{U} \subset G$. Now if $\sigma(V_x,V_y)$ is sufficiently small, then $\overline{V}_y \subset U$ and hence the values $Af(y)$ depend only on values of f on the set $\overline{U}$. From this it follows with the help of approximation theorem Th. 2.3.1 in [3] that it is enough to prove the continuity of Ap at x for any potential p on X.

The assumption (i) implies (see the remark above) the existence of neighbourhoods U, U' of x such that $\overline{U} \subset V_x \subset \overline{V}_x \subset U'$, for which the difference of estimates of $\alpha(V_x,x,p)$ in

$$\alpha(U',x,p) \leqq \alpha(V_x,x,p) \leqq \alpha(U,x,p)$$

is arbitrarily small. For $\sigma(V_x,V_y)$ small enough we have $U \subset V_y \subset U'$ and the analogous estimate together with the continuity of ω gives

$$(13) \qquad |\alpha(V_x,x,p) - \alpha(V_y,x,p)| \longrightarrow 0$$

as $\rho(x,y) \longrightarrow 0$. It is easily seen that for all y sufficiently close to x we have $U \subset V_y$ and hence the functions $\alpha(V_y,\cdot,p)$ are harmonic on U. Moreover, these functions are uniformly bounded for all y from a neighbourhood of x. This implies by Th. 11.1.1 in [3] the equicontinuity of all these functions on a neighbourhood of x. Now from (13), the equicontinuity and the estimate

$$|Ap(x) - Ap(y)| \leqq |\alpha(V_x,x,p) - \alpha(V_y,x,p)| + \\ + |\alpha(V_y,x,p) - \alpha(V_y,y,p)|$$

we shall easily obtain the required continuity of Ap at x.

In the case of classical harmonic functions the mean value over a ball can be interpreted as an "infinite combination" of harmonic measures corresponding to smaller balls with the same centre. In what follows, we shall study A-harmonicity for $A = \{\alpha_x\}_{x \in G}$ with measures α_x of a similar type.

8. __Lemma__. Let D be an open subset of a metric space. Let us suppose:

(i) $\{f_x; x \in D\}$ is a set of positive decreasing and uniformly bounded functions defined on $[0,d]$, $d>0$;

(ii) for any $x \in D$ and $s,t,t' \in [0,d]$, $t<s<t'$, there is a neighbourhood $U(x)$ of x in D such that the inequality

$$f_x(t) \geqq f_y(t) \geqq f_x(t') \tag{14}$$

holds for every $y \in U(x)$;

(iii) g is a continuous strictly increasing function on $[0,d]$, $g(0)=0$.

Then the average

$$\mathcal{A}(f_x,g,c) := (g(c))^{-1} \int_0^c f_x(t)\,dg(t) \tag{15}$$

is a continuous function (w. r. t. the usual product topology) of (x,c) on $D \times]0,d[$.

Proof. Clearly, $c \mapsto \mathcal{A}(f_x,g,c)$ is a continuous function on $]0,d[$ for any $x \in D$. It is easily seen that it is also decreasing. For a fixed $c \in]0,d[$ the continuity of $x \mapsto \mathcal{A}(f_x,g,c)$ follows from the approximation of the integral in (15) by an integral sum combined with the property (14). Indeed, for $P \equiv \equiv \{0 = t_0 < t_1 < \ldots < t_k = c\}$ and $s_i \in]t_{i-1},t_i[$ we can choose a neighbourhood $U(x)$ of x such that $f_x(t_{i-1}) \geqq f_y(s_i) \geqq f_x(t_i)$ holds for all $y \in U(x)$. Now the expression

$$\left|\sum_{i=1}^{k} f_x(t_i)[g(t_i) - g(t_{i-1})] - \sum_{i=1}^{k} f_y(s_i)\,[g(t_i) - g(t_{i-1})]\right|$$

may be done arbitrarily small by the choice of P and s_i while both integral sums are close enough to the corresponding integrals. (For details see the proof of L.11 in [13].) From the separate continuity in x and c and monotonicity in c it follows that $\mathcal{A}(f_x,g,c)$ is continuous on $D \times]0,d[$.

9. <u>Proposition</u>. Let $G \subset X$ be an open set. Let us suppose that

$$\Omega : G \times G \longrightarrow R^1, \quad d : G \longrightarrow R^1, \quad g : [0, \infty[\longrightarrow R^1$$

are positive continuous functions enjoying properties:

(i) $\Omega(x,x) = 0$ for every $x \in G$, $\Omega(x,y) > 0$ for every $x,y \in G$, $x \neq y$;

(ii) for every $x \in G$, $0 < c < d(x)$, the closure of the set

(16) $$V_{x,c} = \{ y \in G ; \Omega(x,y) < c \}$$

is a compact subset of G and its distance (w.r.t. the metric ρ fixed above) from $CV_{x,c'}$ is strictly positive for every $c < c' < d(x)$;

(iii) g is a continuous strictly increasing function and $g(0) = 0$

Then for every $h \in \mathcal{C}(G)$

(17) $$\mathcal{a}h(x,c) := (g(c))^{-1} \int_0^c \alpha(V_{x,t}, x, h)\, dg(t)$$

defines a continuous function of (x,c) on the set $\{(x,c); x \in G, 0 < c < d(c)\}$.

Proof. Fix a point (x,c) in the set and choose a neighbourhood D of x and an interval $]a,b[$ containing c in such a way that $\overline{V}_{x,e} \subset G$ for all $(y,e) \in D \times]a,b[$.

Likewise in the proof of Pr. 7 it suffices to prove the continuity of $\mathcal{a}p$ for every potential p. We shall show that in this case Pr. 9 follows easily from L.8. Indeed, the assumption (iii) implies that L.8(iii) holds for g on $[0,b]$. On the same interval, the functions $f_x : t \mapsto \alpha(V_{x,t}, x, p)$, $x \in D$, satisfy L.8(i).

If $x \in D$ and $s,t,t' \in [0,b]$, $t < s < t'$, then it follows from (ii) and the uniform continuity of Ω on compact subsets of $G \times G$ that there is a neighbourhood $U(x)$ of x in D such that we have $V_{x,t} \subset V_{y,s} \subset V_{x,t'}$ for every $y \in U(x)$. Now (14) is a consequence of the monotonicity of the balayage. Since we have verified that all assumptions of L.8 are fulfilled, the proposition is proved.

10. <u>Corollary</u>. Assume the same as in Pr. 9 and suppose that $\delta \in \mathcal{B}(G)$, $0 < \delta(x) < d(x)$ for every $x \in G$. Then

(18) $$A_1 h(x) := \mathcal{a} h(x, \delta(x)), \qquad x \in G,$$

where $\mathcal{a} h$ is defined by (17), and

(19) $$A_2 h(x) := \alpha(V_{x,\delta(x)}, x, h), \qquad x \in G$$

are Borel operators (on $\mathcal{B}_b(G)$). If δ is moreover continuous, then $A_1 h \in \mathcal{C}_b(G)$ and $A_2 h \in \mathcal{B}_b(G)$ whenever $h \in \mathcal{C}_b(G)$.

Indeed, if $h \in \mathcal{C}_b(G)$ and δ is continuous, then Pr. 9 implies that $\mathcal{a} h(x, \delta(x))$, $x \in G$ is continuous. Put $g(t) = t$, $t > 0$. It is easily seen that

$$A_2 h(x) = \lim_{m \to \infty} n \left(\mathcal{a} h(x, \delta(x)) - \mathcal{a} h(x, \delta(x) - n^{-1}) \right)$$

and hence $A_2 h \in \mathcal{B}_b(G)$. This gives the first part of the assertion.

Now we can give a more explicit description of a broad class of operators A which we met in connection with A-harmonicity in Th. 6. More precisely, we shall show how to construct for the operators A_1, A_2 from (18) and (19) a corresponding L-operator B for which (9) holds provided "the values of the size function δ are not too small" in G. The same will be shown for an operator A from Pr. 7.

11. **Theorem.** Let $G \subset X$ be an open relatively compact set and ρ the metric fixed above.

(a) Assume, moreover, the same on Ω, d, g and σ as what was supposed in Pr. 9 and Cor. 10. If for every compact set $C \subset G$

(20) $$\inf\{\sigma(x); x \in C\} = i_C > 0,$$

then the operators A_1, A_2 defined by (18) and (19) have all properties required in Th. 6.

(b) Assume now the same on ω and V_x what was supposed in Pr. 7. If

(21) $$\sigma(x) = \operatorname{dist}(x, CV_x)$$

fulfils the condition (20), then the operator A from Pr. 7 has all properties required in Th. 6.

Proof. All what remains to prove is the existence of a corresponding operator B. In the first case (i.e. (a)) choose compact sets C_k, $C_k \nearrow G$ and for $k = 1,2,\dots$ put $i_k := (k+1)^{-1} i_{C_k}$ (cf. (20)). Then choose $f_k \in \mathscr{C}_c(X)$ in such a way that $0 \leqq f_k \leqq$ $\leqq i_k - i_{k+1}$, $f_k(x) = i_k - i_{k+1}$ on C_k and $f_k(x) = 0$ on CC_{k+1}. Now denote by Δ the restriction of $\sum f_k$ on G; since $\Delta \in \mathscr{C}(G)$, $0 < \Delta(x) < \sigma(x)$, B can be defined by

$$Bh(x) := \mathcal{a}h(x, \Delta(x)), \qquad x \in G$$

(and, formally, $Bh(x) = h(x)$, $x \in \partial G$) and every $h \in \mathscr{C}(\bar{G})$. Inequality (9) is a consequence of monotonicity of $\mathcal{a}h(x,c)$ from (17) in the variable c.

In case of (b) the construction can be repeated with $\Omega(x,y) = \rho(x,y)$, $d(x) = \mathrm{dist}(x, CG)$ and $g(t) = t$, $t > 0$. Function σ is defined by (21). Inequality (9) now follows from the monotonicity of the balayage.

12. Remarks and comments

(1) Lebesgue in [7] replaced essentially the Laplace equation in the Dirichlet problem by the equation $Af = f$ with a "Lebesgue's operator" A. He obtained one of the first theorems on the restricted mean value property (cf. also [8]).

(2) Our results are a generalization of a theorem proved in the classical case in [11]; the authors use a probabilistic approach. Remark that in R^n the mean values of h w.r.t. balls can be obtained if we choose $\Omega(x,y) = \rho(x,y)$, the Euclidean metric and $g(t) = t^{n-1}$, $t > 0$; then (17) determines the mean value since in the case $\alpha(V_{x,t}, x, \cdot)$ is the normalized surface measure on the sphere $\partial V_{x,t}$.

(3) For the generalization of the Lebesgue's approach to the Dirichlet problem in [7] see [13]. In the formula (20) in [13] instead of $\varepsilon_x^{CD_t}$ should be $\varepsilon_x^{CD_x(t)}$.

(4) In our assertions we used mean values of a quite general form but we admit only functions with a "good boundary behaviour a.e. on ∂G". It would be interesting to know whether similar results can be obtained in the frame of an axiomatics for functions with more general boundary behaviour.

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