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A NOTE ON ISOMORPHIC VARIETIES

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Abstract: We shall characterize all the pairs (Δ, Γ) of similarity types such that the variety of all Δ-algebras is isomorphic (as a category) to some variety of Γ-algebras.

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McKenzie [1] proved that for any finite type Δ, the variety of all Δ-algebras is isomorphic to a variety of (2,1)-algebras (algebras with one binary and one unary operation); he asks if the variety of all (2,1)-algebras is isomorphic to some variety of (2)-algebras (i.e. groupoids). The aim of the present paper is to give a negative answer to this question and, more generally, to characterize all the pairs (Δ, Γ) of types such that the variety of all Δ-algebras is isomorphic to some variety of Γ-algebras.

By a type we mean a set of operation symbols; every operation symbol F is associated with a non-negative integer, denoted by n_F and called the arity of F. Let Δ be a type. A Δ-algebra A is determined by a non-empty set (the underlying set of A, denoted also by A) and by an assignment of an n_F-ary operation on the set A to any symbol $F \in \Delta$; this operation will be denoted by F_A.

Let V, W be two varieties and $X \mapsto X^*$ be a functor from the category V into the category W. Following [1], we say that $X \mapsto X^*$ is an isomorphic functor from V to W if every algebra from W is isomorphic to A^* for some $A \in V$, and if $X \mapsto X^*$ induces a bijection of hom(A,B) onto hom(A^*,B^*) for every $A,B \in V$. (It is easy to see that if $A,B \in V$ then $A \simeq B$ iff $A^* \simeq B^*$.) We say that two varieties V, W are isomorphic if there exists an isomorphic functor from V to W.

<u>Lemma 1</u>. Let V, W be two varieties and $X \mapsto X^*$ be an isomorphic functor from V to W. Then:

(1) If $A \in V$ then A is one-element iff A^* is one-element.

(2) If α is a V-morphism then α is injective iff α^* is injective.

(3) If α is a V-morphism then α is surjective iff α^* is surjective.

Proof. A is one-element iff for any $B \in V$ there is exactly one morphism in hom(B,A). α is injective iff it is a monomorphism. α is surjective iff the following is true for all V-morphisms β, γ : if $\alpha = \gamma\beta$ and if γ is injective then γ is an isomorphism.

<u>Lemma 2.</u> Let V, W be two varieties and $X \mapsto X^*$ be an isomorphic functor from V to W. Let $k \geq 1$ be an integer; let P be a V-free algebra of rank k and suppose that P^* is a W-free algebra of rank 1; let $x_1,\dots,x_k$ be free generators of P and let x be a free generator of P^*. For every $a \in V$ we can define a one-to-one mapping ι_A of A^* onto A^k in this way: if $a \in A^*$ then $\iota_A(a) = (\alpha(x_1),\dots,\alpha(x_k))$ where α is the unique morphism

from hom(P,A) with $\alpha^*(x) = a$. If $\beta \in$ hom(A,B) in V, $a \in A^*$ and $\iota_A(a) = (a_1,\dots,a_k)$ then $\iota_B(\beta^*(a)) = (\beta(a_1),\dots,\beta(a_k))$.

Proof. Evidently, it is possible to define a mapping ι_A of A^* into A^k as above. Conversely, define a mapping $\varkappa_A$ of A^k into A^* as follows: if $a_1,\dots,a_k \in A^k$, put $\varkappa_A(a_1,\dots,a_k) = \alpha^*(x)$ where α is the unique morphism from hom(P,A) with $\alpha(x_1) = a_1,\dots,\alpha(x_k) = a_k$. Evidently, the mappings $\varkappa_A \iota_A$ and $\iota_A \varkappa_A$ are both identical, so that ι_A is bijective and $\varkappa_A$ is its inverse. Let $\beta \in$ hom(A,B), $a \in A^*$ and $\iota_A(a) = (a_1,\dots,a_k)$. There is a unique $\alpha \in$ hom(P,A), with $\alpha^*(x) = a$; we have $a_1 = \alpha(x_1),\dots,a_k = \alpha(x_k)$. Now $\beta\alpha \in$ hom(P,B), $(\beta\alpha)^*(x) = \beta^*(a)$ and so $\iota_B(\beta^*(a)) = (\beta\alpha(x_1),\dots,\beta\alpha(x_k)) = (\beta(a_1),\dots,\beta(a_k))$.

Let V, W be two varieties. By an equivalence between V, W we mean an isomorphic functor from V to W commuting with the underlying set functors. (Then this functor induces a bijection between V, W.)

Lemma 3. Let V, W be two varieties and $X \mapsto X^*$ be an isomorphic functor from V to W. Let P be a V-free algebra of rank 1 and suppose that P^* is a W-free algebra of rank 1, too. Then V, W are equivalent.

Proof. It follows easily from Lemma 2.

Corollary. Let V, W be two varieties of idempotent algebras. If V, W are isomorphic then they are equivalent.

Proof. It follows from Lemma 3 and assertion (1) of Lemma 1.

Lemma 4. Let Δ, Γ be two types, let V be the variety of all Δ-algebras and let W be some variety of Γ-algebras; let $X \mapsto X^*$ be an isomorphic functor from V to W. Then there are an integer $k \geq 1$ and an algebra $P \in V$ such that P is a V-free algebra of rank k and P^* is a W-free algebra of rank 1.

Proof. Evidently, there is an algebra $P \in V$ such that P^* is a W-free algebra of rank 1. Let us call an algebra $A \in W$ s-projective in W if for any surjective morphism α in W and any morphism $\beta \in \hom(A,B)$, where B is the end of α, there exists a morphism γ in W with $\beta = \alpha\gamma$. Every W-free algebra is s-projective in W. Hence P^* is s-projective in W and so P is s-projective in V. However, in V every s-projective algebra is V-free (as it is easy to see). Hence P is V-free of rank k for some cardinal number k. Suppose k=0. Then for every $a \in V$, hom(P,A) contains exactly one morphism; but then $\hom(P^*,B)$ contains exactly one morphism for every $B \in W$, which is evidently impossible. Hence $k \geq 1$. Suppose that k is infinite. Then P is the coproduct (in V) of ω copies of P, so that P^* is the coproduct (in W) of ω copies of P^*; thus P^* is a W-free algebra of rank ω. However, this is impossible.

In the following Lemmas 5,6,7,8,9 and 10 let Δ, Γ be two types, let V be the variety of all Δ-algebras and W be some variety of Γ-algebras; let $X \mapsto X^*$ be an isomorphic functor from V to W; let $k \geq 1$ be an integer [illegible] $P \in V$ be an algebra such that P is a V-free algebra of rank [illegible] P^* is a W-free algebra of rank 1. We shall fix free generators $x_1, \ldots, x_k$ of P and a free generator x of P^*. For every $A \in V$ defi[illegible] as in Lemma 2; write ι instead of ι_A. Fu[illegible]er, let us f[illegible] [illegible]e algebra Q with an infinite countable set of free generators $\{x_{i,j};\ 1\le i<\omega,\ 1\le j\le k\}$. The free generators $x_{i,j}$ of Q will be called variables and the elements of Q - terms. Define morphisms $\alpha_i: P\longrightarrow Q$ by $\alpha_i(x_j) = x_{i,j}$. Then Q is a coproduct (in V) of ω copies of P, with canonical morphisms α_i $(1\le i<\omega)$. Consequently, Q^* is a coproduct (in W) of ω copies of P^*, with canonical morphisms α^*_i. Put $y_i = \alpha^*_i(x)$; then Q^* is a W-free algebra with free generators $y_1, y_2, \dots$ and we have $\iota(y_i) = (x_{i,1},\dots,x_{i,k})$. For every $F\in\Gamma$ denote by $(F^{[1]},\dots,F^{[k]})$ the k-tuple $\iota(F_{Q^*}(y_1,\dots,y_{n_F}))$.

__Lemma 5__. Let $I\subseteq\{1,2,\dots\}$ and let $a\in Q^*$ be an element belonging to the subalgebra of Q^* generated by $\{y_i; i\in I\}$. Put $\iota(a) = (a_1,\dots,a_k)$. Then every variable contained in some of the terms $a_1,\dots,a_k$ belongs to $\{x_{i,j}; i\in I,\ 1\le j\le k\}$.

Proof. There is an endomorphism ε of Q such that $\varepsilon^*(y_i) = y_i$ for all $i\in I$ and $\varepsilon^*(y_i) = y_{i+1}$ for all $i\notin I$. We have $\varepsilon^*(a) = a$ and so $\varepsilon(a_1) = a_1,\dots,\varepsilon(a_k) = a_k$ by Lemma 2; hence $\varepsilon(z) = z$ for any variable z contained in some of the terms $a_1,\dots,a_k$. We have $\varepsilon(x_{i,j}) = x_{i+1,j}$ for all i, j such that $i\notin I$; hence $\varepsilon(x_{i,j}) = x_{i,j}$ implies $i\in I$.

__Lemma 6__. If Γ contains a nullary symbol then Δ contains a nullary symbol.

Proof. It follows from Lemma 5.

__Lemma 7__. Let M be a subset of Q such that every variable belongs to M, the terms $F^{[1]},\dots,F^{[k]}$ belong to M for any symbol $F\in\Gamma$ and $\varepsilon(M)\subseteq M$ for any endomorphism ε of Q mapping all variables into M. Then M=Q.

Proof. Denote by D the set of all $u \in Q^*$ such that if $\iota(u) = (u_1,\dots,u_k)$ then $u_1,\dots,u_k \in M$. Since $\iota(y_i) = (x_{i,1},\dots,x_{i,k})$ and M contains all variables, we have $\{y_1,y_2,\dots\} \subseteq D$. Let us prove that D is a subalgebra of Q^*. Let $F \in \Gamma$ and $d_1,\dots,d_{n_F} \in D$. Put $e = F_{Q^*}(d_1,\dots,d_{n_F})$, $\iota(d_i) = (d_{i,1},\dots,d_{i,k})$ and $\iota(e) = (e_1,\dots,e_k)$; we have $d_{i,j} \in M$. Denote by ε the endomorphism of Q with $\varepsilon^*(y_1) = d_1,\dots, \varepsilon^*(y_{n_F}) = d_{n_F}$ and $\varepsilon^*(y_i) = y_i$ for $i > n_F$. By Lemma 2 we have $\varepsilon(x_{i,j}) = d_{i,j}$ for $i \le n_F$ and $\varepsilon(x_{i,j}) = x_{i,j}$ for $i > n_F$. We have $\varepsilon^*(F_{Q^*}(y_1,\dots,y_{n_F})) = F_{Q^*}(d_1,\dots,d_{n_F}) = e$ and so $\varepsilon(F^{[1]}) = e_1,\dots, \varepsilon(F^{[k]}) = e_k$. By the properties of M, $\{e_1,\dots,e_k\} \subseteq M$ and so $e \in D$. We have proved that D is a subalgebra of Q^* containing the generators and so $D = Q^*$. Hence for every $u \in Q^*$ we have $\iota(u) \in M^k$; but then $M=Q$.

Lemma 8. Let $F \in \Gamma$ be unary; let $a \in Q^*$ be such that $\iota(F_{Q^*}(a))$ is a sequence of pairwise different variables. Then $\iota(a)$ is a sequence of pairwise different variables.

Proof. Put $\iota(F_{Q^*}(a)) = (z_1,\dots,z_k)$ and $\iota(a) = (a_1,\dots,a_k)$. Let ε be an endomorphism of Q with $\varepsilon^*(y_1) = a$, so that $\varepsilon(x_{1,1}) = a_1,\dots, \varepsilon(x_{1,k}) = a_k$. We have $\varepsilon^*(F_{Q^*}(y_1)) = F_{Q^*}(a)$ and so $\varepsilon(F^{[1]}) = z_1,\dots, \varepsilon(F^{[k]}) = z_k$. From this it follows that $F^{[1]},\dots,F^{[k]}$ is a sequence of pairwise different variables; by Lemma 5, $\{F^{[1]},\dots,F^{[k]}\} = \{x_{1,1},\dots,x_{1,k}\}$. Since $\varepsilon(F^{[1]}),\dots, \varepsilon(F^{[k]})$ are pairwise different variables, the same must be true for $\varepsilon(x_{1,1}),\dots, \varepsilon(x_{1,k})$, i.e. for $a_1,\dots,a_k$.

Lemma 9. Let $k \ge 2$. Then there is a symbol $F \in \Gamma$ of arity ≥ 2 such that $F^{[1]},\dots, F^{[k]}$ are pairwise different variables.

Proof. There is an element $a \in Q^*$ with $\iota(a) = (x_{1,1}, \ldots, x_{k,1})$. By Lemma 5, a does not belong to the subalgebra of Q^* generated by y_i, for any i. From this it follows that there are a symbol $F \in \Gamma$ of some arity $n \geq 2$, elements $a_1, \ldots, a_n \in Q^*$ and unary symbols $H^1, \ldots, H^m$ $(m \geq 0)$ such that $a = H^1_{Q^*} \ldots H^m_{Q^*} F_{Q^*}(a_1, \ldots, a_n)$. Put $b = F_{Q^*}(a_1, \ldots, a_n)$. By Lemma 8, $\iota(b)$ is a sequence of pairwise different variables. There is an endomorphism ε of Q with $b = \varepsilon^*(F_{Q^*}(y_1, \ldots, y_n))$; hence $\varepsilon(F^{[1]}), \ldots, \varepsilon(F^{[k]})$ is a sequence of pairwise different variables, so that $F^{[1]}, \ldots, F^{[k]}$ are pairwise different variables.

Lemma 10. There is a mapping $\lambda : \Delta \longrightarrow \Gamma$ with the following three properties:

(1) $n_G \leq k n_{\lambda(G)}$ for all $G \in \Delta$.

(2) If $G_1, \ldots, G_m \in \Delta$ are pairwise different and $\lambda(G_1) = \ldots = \lambda(G_m)$ then $m \leq k$.

(3) If $k \geq 2$ then the set $\Gamma \setminus \lambda(\Delta)$ contains an at least binary symbol.

Proof. Let $G \in \Delta$. Suppose that there is no symbol $H \in \Gamma$ such that $G(z_1, \ldots, z_{n_G}) \in \{H^{[1]}, \ldots, H^{[k]}\}$ for ~~some~~ pairwise different variables $z_1, \ldots, z_{n_G}$. Then the set M of terms which are not of the form $G(z_1, \ldots, z_{n_G})$ with $z_1, \ldots, z_{n_G}$ pairwise different variables satisfies evidently the assumptions of Lemma 7, so that M=Q by Lemma 7, evidently a contradiction. This shows that for every $G \in \Delta$ we can choose some $\lambda(G) \in \Gamma$ such that $G(z_1, \ldots, z_{n_G}) \in \{\lambda(G)^{[1]}, \ldots, \lambda(G)^{[k]}\}$ for some pairwise different variables $z_1, \ldots, z_{n_G}$. (1) follows from Lemma 5, (2) is evident and (3) follows from Lemma 9.

__Theorem 1.__ Let Δ , Γ be two types and let $k \geq 1$ be an integer. The following two conditions (I),(II) are equivalent:

(I) There exists an isomorphic functor $X \mapsto X^*$ from the variety of all Δ-algebras to some variety of Γ-algebras such that for some $P \in V$, P is a V-free algebra of rank k and P^* is a W-free algebra of rank 1.

(II) There exists a mapping $\lambda : \Delta \longrightarrow \Gamma$ such that the following four conditions are satisfied:

(1) $n_G \leq k n_{\lambda(G)}$ for all $G \in \Delta$.

(2) If $G_1, \dots, G_m \in \Delta$ are pairwise different and $\lambda(G_1) = \dots = \lambda(G_m)$ then $m \leq k$.

(3) If $k \geq 2$ then the set $\Gamma \setminus \lambda(\Delta)$ contains an at least binary symbol.

(4) If Γ contains a nullary symbol then Δ contains a nullary symbol.

Proof. The direct implication follows from Lemmas 10 and 6. Now let (II) be satisfied. Denote by V the variety of all Δ-algebras. If k=1 then λ is injective and $n_G \leq n_{\lambda(G)}$ for all $G \in \Delta$; this, together with (4), implies that V is equivalent to a variety of Γ-algebras. Let $k \geq 2$. By (3) there exists an at least binary symbol $S \in \Gamma \setminus \lambda(\Delta)$, and evidently it is enough to consider the case when S is binary. For every $F \in \Gamma$ fix a finite sequence μ_F, consisting of all pairwise different symbols $G \in \Delta$ with $F = \lambda(G)$. If Γ contains nullary symbols, fix a nullary symbol $H \in \Delta$. For every Δ-algebra A define a Γ-algebra A^* with the underlying set A^k as follows:

$S_{A^*}((a_1, \dots, a_k), (b_1, \dots, b_k)) = (b_k, a_1, \dots, a_{k-1})$;

if $F \in \Gamma \setminus \{S\}$ is a symbol of arity $n \geq 1$ and $\mu_F = (G^1, \dots, G^m)$, put

$$F_{A^*}((a_1,\dots,a_k),\ (a_{k+1},\dots,a_{2k}),\dots,(a_{nk-k+1},\dots,a_{nk})) =$$
$$= (G^1_A(a_1,\dots,a_{n_{G_1}}),\dots,G^m_A(a_1,\dots,a_{n_{G_m}}),a_1,\dots,a_1);$$

if $F \in \Gamma$ is nullary and $\mu_F(G^1,\dots,G^m)$, put

$$F_{A^*} = (G^1_A,\dots,G^m_A,H_A,\dots,H_A).$$

For every Δ-morphism $\alpha: A \longrightarrow B$ define a Γ-morphism $\alpha^*: A^* \longrightarrow B^*$ by $\alpha^*(a_1,\dots,a_k) = (\alpha(a_1),\dots,\alpha(a_k))$. It is not difficult to prove that the class W of Γ-algebras isomorphic to A^* for some $A \in V$ is a variety and that $X \longmapsto X^*$ is an isomorphic functor from V to W such that the V-free algebra of rank k corresponds to the W-free algebra of rank 1. We shall not give here a detailed proof of this fact, since it is analogous to that of Theorem 1.1 of [1].

Theorem 2. Let Δ, Γ be two types. For every integer $i \geq 0$ put $d_i = \mathrm{Card}\{F \in \Delta; n_F \geq i\}$ and $g_i = \mathrm{Card}\{F \in \Gamma; n_F \geq i\}$. The variety V of all Δ-algebras is isomorphic to some variety of Γ-algebras iff the following seven conditions are satisfied:

(1) If d_0 is infinite then $d_0 \leq g_0$.

(2) If d_1 is infinite then $d_1 \leq g_1$.

(3) $\mathrm{Min}(d_i; i \geq 0) \leq \mathrm{Min}(g_i; i \geq 0)$.

(4) If $g_2 = 0$ then $d_i \leq g_i$ for all i.

(5) If $g_1 = 1$ then either $d_i \leq g_i$ for all i or $d_1 = 0$.

(6) If $g_0 = 1$ then $d_0 \leq 1$.

(7) If Γ contains a nullary symbol then Δ contains a nullary symbol.

Proof. By Lemma 4, the isomorphism of V to some variety of Γ-algebras is equivalent to the existence of an integer $k \geq 1$ satisfying the condition (I) of Theorem 1 and thus to the existence of k and λ satisfying the condition (II) of Theorem 1. It is not difficult to re-formulate this condition in terms of the cardinal numbers d_i and g_i.

R e f e r e n c e

[1] R. McKENZIE: A new product of algebras and a type reduction theorem (to appear).

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