COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

23,3 (1982)

A NOTE ON CHOOSABILITY IN PLANAR GRAPHS

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Abstract: We call a graph k-choosable if for every assignment of a list of k colors to each vertex, the graph can be properly colored so that each vertex is colored with one of the colors on its list. Erdös, Rubin and Taylor have conjectured that every planar graph is 5-choosable. In this note we show that in a minimal counter-example to this conjecture every vertex of degree five must have a neighbor of degree at least seven.

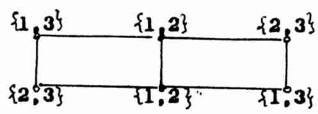
Key words: Choosability, coloring, planar graph.

Classification: 05C15

In [1] Erdös, Rubin and Taylor developed the idea of choosability in graphs. Suppose each vertex of graph G has assigned to it a list of k colors. We say that G is k-choosable or can be k-list-colored if for every assignment of lists, G has a proper coloring with each vertex assigned a color on its list.

We call the minimum k for which G is k-choosable the list-chromatic number of G. It is immediate that the list-chromatic number of G is at least as great as the chromatic number.

That this inequality may be strict is shown by the following example:



The graph is 2-chromatic but the assignment of lists shown in the diagram shows that it is not 2-choosable. It might be noted that to show that a particular graph is k-colorable one needs only to exhibit a k-coloring; to show that it is k-choosable one must show that it can be properly colored from any assignment of lists to the vertices.

It is clear that any planar graph is 6-choosable. The proof is by induction. Delete from the graph a vertex x of degree at most five, and 6-list-color the remaining graph.

When x is restored, it has at most five neighbors already colored, so there must be at least one of the six colors on the list for x that is not used for any of the neighbors. Therefore x can be colored with that sixth color.

In [1] the conjecture is made that every planar graph is 5-choosable.

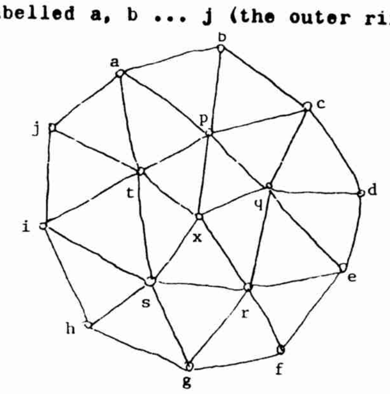
It appears that the techniques used to attack the four color problem cannot be applied to the conjective of five-choosability. Kempe chains for instance cannot be used because the step of recoloring a vertex, say from red to blue, requires that blue be on the list for that vertex.

It is clear that a minimal counter-example to his conjecture can have no vertex of degree four or less. The proof mimics that for the 6-choosability.

In this note we show that in a minimal counter-example to the conjecture every vertex of degree five must have a neighbor of degree at least seven.

The proof is by contradiction. Suppose in a minimal counter-example G we have a vertex x of degree five, all of whose neighbors are of degree six or less. It clearly suffices to consider only the case that all neighbors of x have degree six.

We then have the configuration shown with the five neighbors of x labelled p, q, r, s, t (the inner ring) and their neighbors labelled a, b ... j (the outer ring)



As a matter of notation: let L(v) denote the set of colors on the list of v; let k(v) be the color assigned to v in a coloring of G. Say L(v) = {1,2,3,4,5}.

Delete x and 5-list-color the remaining graph. When x is restored it will be possible to color x unless the five vertices in the inner ring have been colored using all five of the colors in L(x).

Without loss of generality, say p, q, r, s, t are colored 1, 2, 3, 4, 5 respectively. If p can be re-colored with a color other than 1, then x can be colored 1. We cannot re-color p with any color used for a, b, or c but there must be at least one color on L(p) other than 1, k(a), k(b), k(c).

If this color is neither 2 nor 5, then re-color p with this color. Then x can be colored 1.

Say therefore without loss of generality L(p) = {1,2, k(a), k(b),k(c)}. Recolor p with color 2. Now q must be re-colored. L(q) must have a fifth color other than 2, k(c), k(d), k(e). If this color is other than 3, then re-color q with this color and color x with 2.

Say therefore that L(q) = {2,3, k(c),k(d),k(e)}. Repeating the above procedure with vertex r, then s, then t we see that the only obstruction to coloring x occurs if L(r) = {3,4, k(e), k(f),k(g)}; L(s) = {4,5,k(g),k(h),k(i)}, L(t) = {5,1,k(i),k(j), k(a)}.

To look at this another way, we see that if x and the entire ring p , q, r, s, t are all deleted and the remaining graph is 5-list colored, then the only obstruction to restoring x, p, q, r, s, t and extending the coloring to them is if the lists for the inner ring and the coloring of the outer ring are as described above.

We now propose two distinct methods of reduction, which one to be used depending on the lists for the inner ring. In each case the reduction will make impossible the coloring of the outer ring in a manner such that the coloring cannot be extended to a coloring of G.

Case i: The lists for the vertice in the inner ring have in their union at least ten colors, including therefore at least five colors other than those in L(x); say they include 6, 7, 8, 9, 10.

Define graph G' by deleting vertices x, p, q, r, s, t and replacing them with a new vertex x' that is joined to each vertex a, b, c ... j of the outer ring. Assign to x' the list $L(x') = \{6,7,8,9,10\}$. By induction G' can be 5-list colored; say with x' colored 6. But since x' is adjacent in G' to each vertex of the outer ring, this gives a coloring with none of the vertices of the outer ring colored 6. Since 6 is on the list of one of the vertices of the inner ring, this list coloring can be extended in to x, p, q, r, s, t.

Case ii: The lists for the vertices in the inner ring have in their union at most nine colors. Then we must have some two of the ten colors k(a),k(b) ... k(j) the same. Say $k(v) = k(w)$ for some $v,w \in \{a,b,...,j\}$.

Define graph G' by deleting x, p, q, r, s, t and joining vertex v to each other vertex of the outer ring. By induction G' can be 5-list-colored. In this coloring no other vertex of the outer ring can be colored the same as v. Therefore this coloring can be extended in to color x, p, q, r, s, t.

R e f e r e n c e

[1] P. ERDÖS, A.L. RUBIN, H. TAYLOR: Choosability in Graphs, Proc. West Coast Conference on Combinatorics, Graph Theory, and Computing, Arcala, Calif., 1979 (Utilitas Math. Publ. Inc., Winnipeg 1980), 125-157.

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(Oblatum 30.4. 1982)

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

23,3 (1982)

MODEL-THEORETIC PROPERTIES OF CAUSE-AND-EFFECT STRUCTURES

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Abstract: Some questions of axiomatizability and decidability connected with the study of so-called cause-and-effect structures (as introduced by me under the influence of von Wright) are treated.

Key words: Causality relation, axiomatizability, decidability.

Classification: 03A05, 03B25, 03C20

Let a cause-and-effect structure be defined as follows. The domain consists of a set T of moments and a set S of states; the elements of $T\times S$ are called events. As relations and functions we have a chronological order $<\ \subseteq T\times T$, a time addition $+:T\times T\longrightarrow T$, a possibility of events $\Diamond\subseteq T\times S$, an actuality of events $\Box\subseteq T\times S$, and a cause-and-effect relation $\mapsto\ \subseteq T\times S\dot{\times} T\times S$ (we write $t,s\mapsto t',s'$ instead of $\mapsto(t,s,t',s')$). The axioms we assume to be fulfilled by cause-and-effect structures are

(1) $\langle T,<,+\rangle$ is an ordered abelian group

(2) $\forall t\exists s\ \Diamond(t,s)$

(3) $\forall t\exists!!\ \Box(t,s)$

(4) $\forall t,s(\Box(t,s)\longrightarrow\Diamond(t,s))$

(5) $\forall t_1,s_1,t_2,s_2((t_1,s_1\mapsto t_2,s_2)\wedge\Box(t_1,s_1)\longrightarrow\Box(t_2,s_2))$

(6) $\forall t_1,s_1,t_2,s_2((t_1,s_1 \mapsto t_2,s_2) \longrightarrow t_1 < t_2)$

(7) $\forall t_1,s_1,t_2,s_2,t(((t_1,s_1 \mapsto t_2,s_2) \longleftrightarrow$

$\longleftrightarrow (t_1 + t,s_1 \mapsto t_2 + t,s))$

(8) $\forall t,s(\Diamond (t,s) \longrightarrow \exists t',s'(\Diamond (t',s') \wedge (t',s' \mapsto t,s)))$.

Let CES denote the class of all cause-and-effect structures.

Cause-and-effect structures differ from causality structures as introduced (under inspiration of [1]) in [2] in that the axiom (5) of [2] constating, intuitively spoken, that "the behaviour of the system in the past is uniquely determined" is missing.

Given $\mathfrak{A} = \langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle \in$ CES, there is a natural way of embedding $\mathfrak{A}$ into a causality structure $\mathfrak{A}'$ by proceeding as follows. Let $\mathfrak{A}' = \langle (T \times T) \cup (S \cup \{s_0\}), <', +', \Diamond', \mapsto', \Box' \rangle$, where

$$<' = \{\langle\langle t_1,t_2\rangle, \langle t_3,t_4\rangle\rangle : t_1 < t_3 \vee (t_1 = t_3 \wedge t_2 < t_4)\}$$

$$+' = \{\langle\langle t_1,t_2\rangle, \langle t_3,t_4\rangle, \langle t_5,t_6\rangle\rangle : t_1 + t_3 = t_5 \wedge t_2 + t_4 = t_6\}$$

$$\Diamond' = \{\langle\langle t_1,t_2\rangle,s\rangle : (t_1 = 0 \wedge \Diamond(t_2,s)) \vee (t_1 \neq 0 \wedge s = s_0)\}$$

$$\mapsto' = \{\langle\langle t_1,t_1'\rangle,s_1,\langle t_2,t_2'\rangle,s_2\rangle : (t_1 = t_2 = 0 \wedge t',s \mapsto t',s) \vee (t_1 \neq 0 \wedge t_2 \neq 0 \wedge t_1 < t_2)\}$$

$$\Box' = \{\langle\langle t_1,t_2\rangle,s\rangle : (t_1 = 0 \wedge \Box(t_2,s)) \vee (t_1 \neq 0 \wedge s = s_0)\}.$$

$\Box'$ is obtained from $\mathfrak{A}$ by adding a one-state (and, hence, uniquely determined) "past" which precedes the whole "world" $\mathfrak{A}$ and (in order to secure (1)) a one-state "futuro" (the same state as in the past) which follows the whole "world" $\mathfrak{A}$. Of course, the "metatheoretical complicatedness" of $\mathfrak{A}'$ is not exceeding that of $\mathfrak{A}$ although the technical treatment of $\mathfrak{A}'$

may be more labourious than that of $\mathfrak{A}$. This gives motivation to investigate the model-theoretic properties of causality structures by investigating the model-theoretic properties of cause-and-effect structures.

Let $\mathfrak{A} = \langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle \in \mathrm{CES}$ be called <u>proper</u>, if, for every $\langle t, s \rangle \in \Diamond$, there is $\Box' \subseteq T \times S \times T \times S$ such that $\langle t, s \rangle \in \Box'$ and, likewise, $\mathfrak{A}' = \langle T \cup S, <, +, \Diamond, \mapsto, \Box' \rangle \in \mathrm{CES}$. The class of proper cause-and-effect structures will be denoted by PCES.

<u>Theorem 1</u>: With respect to the signature $\langle <, +, \Diamond, \mapsto, \Box \rangle$, PCES is not EC.

Proof. We demonstrate CES $\setminus$ PCES not to be closed under the operation of taking ultraproducts.

Let $\mathfrak{A}_n = \langle (\omega^* + \omega) \cup S_n, <, +, \Diamond_n, \mapsto_n, \Box_n \rangle$ $(n \in \omega)$ be defined as follows:

1. $\langle \omega^* + \omega, <, + \rangle$ is isomorphic to the additive group of integers
2. $S_n = \{0,1,2,3\} \times (\omega^* + \omega)$
3. $\Diamond_n = \{\langle x, \langle 0,x \rangle\rangle : x \equiv 0(2) \wedge x \leq 2n\}$
   $\cup \{\langle x, \langle 1,x \rangle\rangle : x \equiv 1(2) \wedge x \leq 2n+1\}$
   $\cup \{\langle x, \langle 2,x \rangle\rangle : x \leq 0 \vee (x \equiv 0(2) \wedge x \leq 2n+2)\}$
   $\cup \{\langle x, \langle 3,x \rangle\rangle : x > 1 \wedge (x \equiv 1(2) \vee x \geq 2n+1)\}$
4. $\mapsto_n = \Diamond_n^2 \cap \{\langle x, \langle y,x \rangle, x', \langle y',x' \rangle\rangle :$
   $: (y = y' = 3 \wedge x' = x+2 \wedge 0 < x \leq 2n+1)$
   $\vee (y = y' = 3 \wedge x' = x+1 \wedge x \geq 2n+1)$
   $\vee (y = y' = 2 \wedge x' = x+1 \wedge x < 0)$
   $\vee (y = y' = 2 \wedge x' = x+2 \wedge 0 \leq x)$

$\vee (y=y'=1 \wedge x'=x+2)$

$\vee (y=y'=0 \wedge x'=x+2$

$\vee (\langle x,y\rangle = \langle 0,2\rangle \wedge \langle x',y'\rangle = \langle 1,3\rangle)$

$\vee (\langle x,y\rangle = \langle 2n+1,1\rangle \wedge \langle x',y'\rangle = \langle 2n+2,2\rangle)$

$\vee (\langle x,y\rangle = \langle 2n+2,2\rangle \wedge \langle x',y'\rangle = \langle 2n+3,3\rangle)$

$\vee (\langle x,y\rangle = \langle 2n,0\rangle \wedge \langle x',y'\rangle = \langle 2n+3,3\rangle)\}$

5. $\Box_n = \Diamond_n \cap (\{\langle x,\langle 0,x\rangle : x < 2n\} \cup$

$\{\langle\langle x,\langle 1,x\rangle\rangle : x < 2n+1\} \cup$

$\{\langle x,\langle 2,x\rangle\rangle : x = 2n+2\} \cup$

$\{\langle x,\langle 3,x\rangle\rangle : x > 2n+2\})$.

$\mathfrak{A}_3$ is illustrated by fig. 1 ($\Box_3$ cannot be taken from the figure itself, but this does not matter).

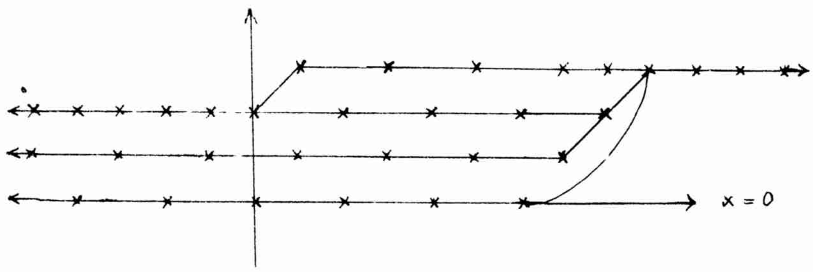

Fig. 1

In this figure, the event $\langle x,\langle y,x\rangle\rangle$ is marked by a cross at the point $\langle x,y\rangle$, and two crosses are connected by an arc if and only if the corresponding events are in cause-and-effect relation (Note that $\mapsto$ is nct transitive!). It is easy to check that each $\mathfrak{A}_n$ is CES (the validity of (5) is based on the fact that states belonging to differenct events are different - a fact which cannot be taken from fig. 1 because $\langle x,\langle y,x\rangle\rangle$ is simply coded by $\langle x,y\rangle$; the validity of the remaining axioms can immediately be seen). On the other hand, no $\mathfrak{A}_n$ is PCES. For instance, there is no $\mathfrak{A}'_n = \langle\{\omega^* + \omega\} \cup S, <, +, \Diamond_n, \mapsto_n, \Box'_n\rangle \in$ CES such that $\langle 2,\langle 0,2\rangle\rangle \in \Box'_n$. For, assuming $\langle 0,\langle 2,0\rangle\rangle \in \Box'_n$, we have, by (5): $\langle 1,\langle 3,1\rangle\rangle, \langle 3,\langle 3,3\rangle\rangle, \ldots, \langle 2n+1,\langle 3,2n+1\rangle\rangle, \langle 2n+2,\langle 3,2n+2\rangle\rangle \in \Box'_n$, and, again by (5), $\langle 2,\langle 2,2\rangle\rangle, \langle 4,\langle 2,4\rangle\rangle, \ldots, \langle 2n+2,\langle 2,2n+2\rangle\rangle \in \Box'_n$, but $\langle 2n+2,\langle 3,2n+1\rangle\rangle \in \Box'_n$, $\langle 2n+2,\langle 2,2n+2\rangle\rangle \in \Box'_n$ is in contradiction with (3).

Next we show that $\prod_{n\in\omega} \mathfrak{A}_n/\mathfrak{U} \in$ PCES, where $\mathfrak{U}$ is a non-principal ultrafilter over ω.

Let us investigate the structure $\prod_{n\in\omega} \mathfrak{A}_n/\mathfrak{U}$. The order is of type $(\omega^* + \omega)\cdot(\tau^* + \tau)$, so that the moments can be coded by couples $\langle \pm\alpha, n\rangle$, where $\alpha \in \tau$, $n \in \omega^* + \omega$. The substructure induced by all events possible in moments of type $\langle 0,n\rangle$ is illustrated by fig. 2:

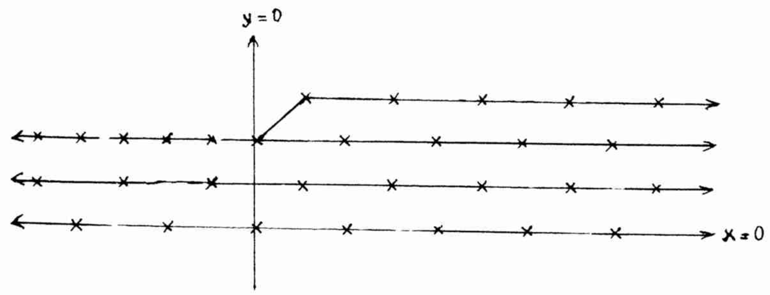

Fig. 2

The substructure induced by all events possible in moments $\langle \alpha_0, n \rangle$ where $\langle \alpha_0, n_0 \rangle$ is the moment attached to the event p of $_{m}\prod_{\in\omega} \mathfrak{A}_n / \mathfrak{U}$ represented by the sequence $\{\langle 2n, \langle 2, 2n \rangle\rangle\}_{n\in\omega}$ is illustrated by fig. 3:

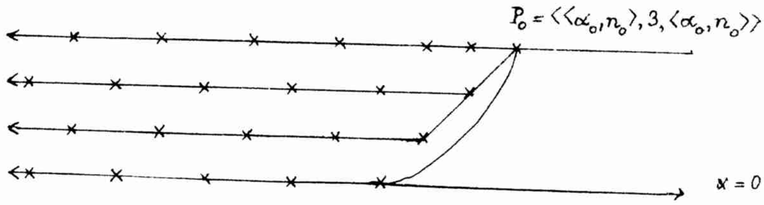

Fig. 3

For moments resting, the corresponding substructures are illustrated by fig. 4 ($\langle \alpha, 0 \rangle < 0$), fig. 5 ($0 < \langle \alpha, 0 \rangle < \langle \alpha_0, 0 \rangle$) and fig. 6 ($\langle \alpha, 0 \rangle > \langle \alpha_0, 0 \rangle$):

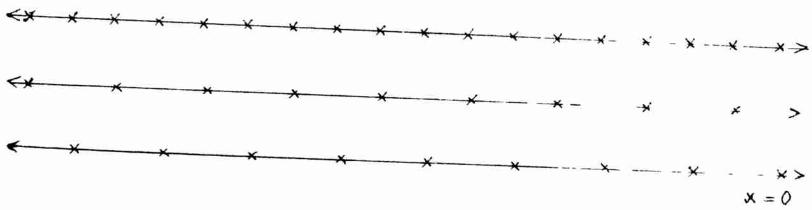

Fig. 4