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THE SPACE OF COMPLETE SUBGRAPHS OF A GRAPH

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Abstract: A remainder of ω is a space X which is homeomorphic to $\gamma\omega - \omega$, for some T_2 compactification $\gamma\omega$ of the countable discrete space ω. It is folklore that all separable T_2 spaces are remainders. We show that in a certain model of ZFC there is a graph G such that its space of complete subgraphs is a compact ccc space of weight at most continuum which is not a remainder. Furthermore, the graph G yields a supercompact Fréchet-Urysohn space with these properties. A modification yields a compact space of size continuum with only one point of non-first-countability that is also not a remainder.

Key words and phrases: Complete subgraph, ccc, remainder, Fréchet-Urysohn.

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1. Introduction. A remainder (of ω) is a space X which is homeomorphic to $\gamma\omega - \omega$, for some T_2 compactification $\gamma\omega$ of the countable discrete space ω. A possible remainder (of ω) is a compact T_2 space of weight at most continuum. All remainders are possible remainders. Which possible remainders are remainders is not sufficiently understood yet.

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I. Parovičenko [6] has proven that all possible remainders of weight at most ω_1 are remainders and hence that all possible remainders are remainders if one assumes the continuum hypothesis CH. On the other side of the coin, K. Kunen [4] has shown that it is consistent with ZFC that ordinal space $\omega_2 + 1$ is a possible remainder that is not a remainder. Other examples of possible remainders that are not remainders are given by E. van Douwen and T. Przymusinski in [2].

It is known that all separable possible remainders are remainders and T. Przymusiński [7] has proven that all perfectly normal possible remainders are remainders. In Section 4, we will show that separable cannot be generalized to ccc by constructing a consistent counterexample. Whether this could be done had been asked in [7]. Our example is also a supercompact Fréchet-Urysohn space. The question of whether every first countable possible remainder is a remainder, cf. [7], is still open, but by modifying our main example, we get a possible remainder that is not a remainder and that has only one point of non-first-countability.

In Section 2, we list the definitions and concepts used in our paper. In Section 3, we investigate the space of all complete subgraphs of a graph. Our main example is a space of this type.

2. Preliminaries. Our set theory notation is standard. A cardinal is an initial ordinal. The first three infinite cardinals are denoted by ω, ω_1 and ω_2. The cardinal of the continuum 2^{ω} is denoted by c. If X is a set, then $\mathcal{P}(X)$ is

the set of all subsets of X. A collection of sets is linked if every two sets in the collection have a non-empty intersection. For a cardinal κ, $[\kappa]^2$ represents the set of all 2-element subsets of κ.

The quotient algebra, $\mathcal{P}(\omega)$ modulo its ideal of finite sets, is denoted by P/F. P/F is isomorphic to the boolean algebra of clopen sets of $\beta\omega - \omega$, the Stone-Čech remainder of ω. As such, if X is a compact 0-dimensional T_2 space which is a remainder of ω, then the boolean algebra of clopen sets of X is embeddable in P/F.

A graph G consists of a set of vertices and undirected edges between some of its pairs of vertices. If there is an edge between vertices v and w, then we write v —— w, if not, then we write v —/— w. A subgraph H of G consists of a subset of vertices and exactly the same edges between them as in the graph G. H is a complete subgraph of G if every two vertices of H are joined by an edge.

If $(P, \leq)$ is a partially ordered set, then a finite subset F of P is compatible if there exists $p \in P$ such that for all $q \in F$, $p \leq q$. If F is not compatible, then we say that F is incompatible. A subset A of P is an antichain if every 2-element subset of A is incompatible. P is ccc if P does not contain an uncountable antichain. P has precaliber κ if every subset R of P of size κ contains a subset S of size κ such that every finite subset of S is compatible.

The weight of a space X is the least cardinal of a base for X. A closed subbase S for a space X is binary if every linked subcollection of S has a non-empty intersection. X is supercompact if X has a binary closed subbase. A space X is <u>ccc</u> if every collection of pairwise disjoint open sets is countable. X is <u>Fréchet-Urysohn</u> if whenever $A \subseteq X$ and $x \in Cl_X A$, then there exists a sequence $\{a_n : n < \omega\} \subseteq A$ such that $(a_n)_{n<\omega}$ converges to x.

3. <u>The space of complete subgraphs of a graph</u>. Let G be an infinite graph. Set $C(G) = \{C: C$ is a complete subgraph of $G\}$. We include the empty set ϕ as a complete subgraph of G. For each $v \in G$, set $v^+ = \{C: C \in C(G)$ and $v \in C\}$ and $v^- = \{C: C \in C(G)$ and $v \notin C\}$. We topologize C(G) by using $\bigcup_{v \in G} \{v^+, v^-\}$ as a closed (also open) subbase. If F is a finite subset of G, we set $F^+ = \bigcap_{v \in F} v^+$ and $F^- = \bigcap_{v \in F} v^-$. If we identify C(G) with $\{f: f$ is a characteristic function of a complete subgraph of $G\}$, then C(G) has the subspace topology inherited from the Tychonov product 2^G. As such, C(G) is a compact T_2 space. For each $n < \omega$, set $F_n(G) = \{C: C \in C(G)$ and $|C| \leq n\}$. Set $F(G) = \bigcup_{n<\omega} F_n(G)$. It is easily seen that each $F_n(G)$ is a closed subspace of C(G), that each $F_n(G) - F_{n-1}(G)$ is discrete, and that F(G) is dense in C(G). As an exercise, the reader may prove that if G is a complete graph, then C(G) is homeomorphic to 2^G and if G is an independent graph, then C(G) is homeomorphic to the one-point compactification of a discrete space of size $|G|$.

<u>Proposition 3.1</u>. C(G) is a supercompact space of weight $|G|$.

Proof: Let $\{v^+: v \in A\} \cup \{v^-: v \in B\}$ be a linked collection. This implies that $A \subset C(G)$ and $A \cap B = \phi$. Hence, $A \in \bigcap_{v \in A} v^+ \cap$

$\cap \bigcap_{v \in B} v^-$. Thus, $\bigcup_{v \in G} \{v^+, v^-\}$ is a binary closed subbase and C(G) is supercompact.

The weight of C(G) is clearly at most $|G|$. Since $\{v^+ : v \in G\}$ is a collection of $|G|$ distinct clopen sets and C(G) is compact, its weight is exactly $|G|$.

If G is countable, then C(G) is a compact metric space. Whereas, if G is uncountable, then the ϕ is not even a G_δ. So, C(G) is first countable iff G is countable. However, we can get non-trivial sequential properties of C(G) for uncountable G.

__Proposition 3.2.__ C(G) is Fréchet-Urysohn iff every complete subgraph of G is countable.

Proof: (only if). Let $A \in C(G)$. $A \in C\ell\{F : F$ is a finite subset of $A\}$. By assumption, there exists a sequence $(F_n)_{n<\omega}$ of finite subsets of A converging to A. But, then $A = \bigcup_{n<\omega} F_n$. For, if $a \in A - \bigcup_{n<\omega} F_n$, then a^+ is a neighbourhood of A disjoint from $\{F_n : n < \omega\}$. Thus, A is countable.

(if). C(G) viewed as $\{f : f$ is a characteristic function of a complete subgraph of $G\}$ is now a subspace of a Σ-product in 2^G which is well-known to be Fréchet-Urysohn.

__Proposition 3.3.__ C(G) is ccc iff F(G), partially ordered by $F \leq K$ iff $K \subseteq F$, is ccc.

Proof: (only if). Let A be an uncountable subset of F(G). $\{F^+ : F \in A\}$ is an uncountable collection of distinct clopen sets of C(G). By assumption, there exists $F \neq K$ in A such that $F^+ \cap K^+ \neq \phi$. Hence $F \cup K \in F(G)$ and $F \cup K \leq F$ and $F \cup K \leq K$.

(if). Let $\{F_\alpha^+ \cap K_\alpha^- : \alpha < \omega_1\}$ be an uncountable collection of distinct non-empty basic open sets of C(G). We must show that there are $\alpha \neq \beta$ such that $(F_\alpha^+ \cap K_\alpha^-) \cap (F_\beta^+ \cap K_\beta^-) \neq \phi$, i.e., that $F_\alpha \cup F_\beta \in F(G)$ and $(F_\alpha \cup F_\beta) \cap (K_\alpha \cup K_\beta) = \phi$. By restricting to an uncountable subcollection, we may as well assume that there exists $n < \omega$ and $m < \omega$ such that for each $\alpha < \omega_1$, $|F_\alpha| = n$ and $|K_\alpha| = m$. Since each $F_\alpha^+ \cap K_\alpha^- \neq \phi$, we know that $F_\alpha \in F(G)$ and that $F_\alpha \cap K_\alpha = \phi$. If there exists $\alpha \neq \beta$ such that $F_\alpha = F_\beta$, then $F_\alpha \cup F_\beta \in F(G)$ and $(F_\alpha \cup F_\beta) \cap$ $\cap (K_\alpha \cup K_\beta) = \phi$ and we are done. So, we assume that $\{F_\alpha :$ $: \alpha < \omega_1\}$ is faithfully indexed. There cannot exist an infinite subset I of ω_1 such that for every α, β in I, either $F_\alpha \cap K_\beta \neq \phi$ or $F_\beta \cap K_\alpha \neq \phi$, as this would force $\sup\{|F_\alpha \cup K_\alpha| : \alpha \in I\} = \omega$. Invoking the partition relation $\omega_1 \longrightarrow (\omega_1, \omega)$, cf. pg. 115 of [3], we conclude that there exists an uncountable $A \subseteq \omega_1$ such that for every α, β in A, $F_\alpha \cap K_\beta = \phi$ and $F_\beta \cap K_\alpha = \phi$. Now, by our assumption, there exists $\alpha \neq \beta$ in A such that $F_\alpha \cup F_\beta \in F(G)$. Since $(F_\alpha \cup F_\beta) \cap$ $\cap (K_\alpha \cup K_\beta) = \phi$, we have proven C(G) to be ccc.

The next proposition is the reason why the space that we construct in Section 4 is not a remainder of ω.

<u>Proposition 3.4.</u> If C(G) is a remainder of ω, then there exists $\varphi : G \longrightarrow \mathcal{P}(\omega)$ such that for all v, w in G, v —— w iff $\varphi(v) \cap \varphi(w)$ is infinite.

Proof: If C(G) is a remainder of ω, then its boolean algebra of clopen sets is embedded in P/F. Let h be such an embedding. Let π be a choice function for P/F, i.e., $\pi(b) \in b$ for all $b \in P/F$. Define $\varphi : G \longrightarrow \mathcal{P}(\omega)$ by $\varphi(v) = \pi(h(v^+))$.

Since v —— w iff $v^+ \cap w^+ \neq \phi$, φ does the job required.

4. <u>The Cohen-generic graph on ω_2 vertices</u>. Our basic reference for the forcing used is K. Kunen's Set Theory [5]. We refer there for all of our undefined notions.

Starting with a partially ordered P in a ground model M, we get a generic filter $G \subseteq P$ in the universe and form a new model M[G] the least model of ZFC containing M and G. There is a forcing language in M involving P and names $\underline{x}$ for all sets x in M[G]. If φ is a formula of set theory, and $p \in P$, then $p \Vdash \varphi(\underline{x_1},\dots,\underline{x_n})$ iff for every generic filter H containing p, M[H] satisfies $\varphi(x_1,\dots,x_n)$. For our purposes, we need only know what a name for an M[G]-subset of ω is. An M-subset $\underline{x}$ of $\omega \times P$ names the following M[G]-subset of ω , $x = \{n$: there exists $s \in G$ with $(n,s) \in \underline{x}\}$. Conversely, every M[G]-subset x of ω has such a name $\underline{x}$. Even more, if x is an M[G]-subset of ω, then x has a nice name of the form $\underline{x} = \bigcup_{n<\omega} \{n\} \times A_n$, where each A_n is an antichain of P.

Let M be our ground model. Set $P = \{p$: p is a finite partial function of $[\omega_2]^2$ into $2\}$. We say that $p \leq q$ if $q \subseteq p$. As a partial order, P is isomorphic to the partial order of basic clopen sets of $2^{[\omega_2]^2}$ under inclusion and thus P is ccc and has precaliber ω_2. Since P is ccc, the cardinals of M[G] are precisely the cardinals of M.

In the universe, let $G \subseteq P$ be a generic filter. In M[G], the model gotten by adding ω_2 Cohen-reals to M, $\omega_2 \leq c$. In M[G], $\bigcup G: [\omega_2]^2 \to 2$. Let G represent the graph on ω_2 described by: α —— β iff $\bigcup G(\{\alpha,\beta\}) = 0$. No confusion will arise from our double use of the letter G.

Theorem 4.1. In $M[G]$, $C(G)$ is a supercompact, ccc, Fréchet-Urysohn space of weight ω_2 and $C(G)$ is not a remainder of ω.

Proof: That $C(G)$ is supercompact and of weight ω_2 follows from Proposition 3.1.

To prove that $C(G)$ is ccc, according to Proposition 3.3, we must show that $F(G)$, ordered by $F \leq K$ iff $K \subseteq F$, is ccc. This is a standard exercise in forcing using a delta system. See problem C6 on page 292 of [5].

To prove that $C(G)$ is Fréchet-Urysohn, according to Proposition 3.2, we must show that every complete subgraph of G is countable. Let A be an uncountable subgraph of G. Consider the dual graph G' of G, defined as follows: α —— β iff $\bigcup G(\{\alpha, \beta\}) = 1$. As in the preceding paragraph, $C(G')$ is ccc. Therefore, in $C(G')$, there exists $\alpha \neq \beta$ in A such that $\alpha^+ \cap \beta^+ \neq \phi$. This means that α —— β in G' and hence α —/— β in G.

To prove that $C(G)$ is not a remainder of ω, according to Proposition 3.4, it suffices to show that if $\varphi : \omega_2 \longrightarrow \mathcal{P}(\omega)$, then there exists $\alpha \neq \beta$ such that either α —— β and $\varphi(\alpha) \cap \varphi(\beta)$ is finite or α —/— β and $\varphi(\alpha) \cap \varphi(\beta)$ is infinite. To do this, we will take a $p \in P$ that forces our hypothesis (with names) and find a $q \leq p$ that forces our conclusion (with names).

We work in M now. Let $p \Vdash \underline{\varphi} : \omega_2 \longrightarrow \underline{\mathcal{P}(\omega)}$. For each $\alpha < \omega_2$, choose $p_\alpha \leq p$ such that $p_\alpha \Vdash \underline{\varphi}(\alpha) = \underline{x_\alpha}$, where $\underline{x_\alpha}$

is a nice name for a subset of ω. That is, for each $\alpha < \omega_2$, $\underline{x_\alpha} = \bigcup_{n<\omega} \{n\} \times A_n^\alpha$, where each A_n^α is an antichain of P. Since P is ccc, for each $\alpha < \omega_2$, $\underline{x_\alpha}$ is a countable set. Since P has precaliber ω_2, we now choose $D \subseteq \omega_2$ of size ω_2 such that for every α, $\beta \in D$, p_α and p_β are compatible, i.e., $p_\alpha \cup p_\beta \in P$. For each $\alpha \in D$, set $D_\alpha = \{\gamma < \omega_2 : \gamma$ is mentioned in p_α or in $\underline{x_\alpha}\}$. $\{D_\alpha : \alpha \in D\}$ is a collection of ω_2 countable sets. Invoking Hajnal's Free-set theorem cf. page 96 of [3], we can get $\alpha \neq \beta$ in D such that $\alpha \notin D_\beta$ and $\beta \notin D_\alpha$.

Set $t = p_\alpha \cup p_\beta \cup \{(\{\alpha, \beta\}, 1)\}$. If $t \Vdash \underline{x_\alpha} \cap \underline{x_\beta}$ is infinite, then let $q = t$ and we have $q \leq p$ and $q \Vdash \alpha \not\perp \beta$ and $\underline{x_\alpha} \cap \underline{x_\beta}$ is infinite. So, we are finished. If not, then there exists $r \leq t$ such that $r \Vdash \underline{x_\alpha} \cap \underline{x_\beta}$ is finite. Consider the following automorphism h of P that only affects edges between α and β: Let $p \in P$. Set $\mathrm{dom}(h(p)) = \mathrm{dom}(p)$ and if $\{\gamma, \delta\} \in \mathrm{dom}\ p$ define $h(p)(\{\gamma, \delta\})$ to be $p(\{\gamma, \delta\})$ if $\{\gamma, \delta\} \neq \{\alpha, \beta\}$ and to be $1 - p(\{\alpha, \beta\})$ if $\{\gamma, \delta\} = \{\alpha, \beta\}$.

__Claim:__ $h(r) \Vdash \underline{x_\alpha} \cap \underline{x_\beta}$ is finite.

Proof of Claim: Let H be a generic filter of P containing $h(r)$. Then $h(H) = \{h(s) : s \in H\}$ is a generic filter of P containing $h(h(r)) = r$. Since $r \Vdash \underline{x_\alpha} \cap \underline{x_\beta}$ is finite, $\{n < \omega :$ there exists $s \in h(H)$ with $(n,s) \in \underline{x_\alpha}\} \cap \{n < \omega :$ there exists $s \in h(H)$ with $(n,s) \in \underline{x_\beta}\}$ is finite. But $h(H)$ and H have precisely the same s's such that $(n,s) \in \underline{x_\alpha} \cup \underline{x_\beta}$ since for no $n < \omega$ and for no s with $\{\alpha, \beta\} \in \mathrm{dom}\ s$, is $(n,s) \in \underline{x_\alpha} \cup \underline{x_\beta}$. Consequently, $\{n < \omega :$ there exists $s \in H$ with $(n,s) \in \underline{x_\alpha}\} \cap \{n < \omega :$ there exists $s \in H$ with $(n,s) \in \underline{x_\beta}\}$ is finite. We have proven the claim.

In this case, let $q = h(r)$ and we have $q \leq p$ and $q \Vdash \alpha — \beta$ and $\underset{\sim}{x}_\alpha \cap \underset{\sim}{x}_\beta$ is finite.

We now present two byproducts of this example.

Example 4.2. In $M[G]$, $F_2(G)$ is a possible remainder of size ω_2 which is a union of 3 discrete subspaces but which is not a remainder.

Proof: $F_2(G)$ is not a remainder because $v^+ \cap w^+ \neq \phi$ iff $v^+ \cap w^+ \cap F_2(G) \neq \phi$. Also, $F_2(G) = [F_0(G)] \cup [F_1(G) - F_0(G)] \cup \cup [F_2(G) - F_1(G)]$, each of which is discrete. We remark that 3 is the least possible number here since a possible remainder which is the union of 2 discrete subspaces is just a finite disjoint union of one point compactifications of discrete spaces and hence is a remainder.

Example 4.3. In $M[G]$, there is a first countable, locally compact space of size c no compactification of which is a remainder. In particular, its one-point compactification is not a remainder.

Proof: Let $h: \omega_2 \to 2^\omega$ be an injection. Set $X = [\omega_2 \times 2^\omega] \cup \cup [F_2(G) - F_1(G)]$. We define a countable neighbourhood base of clopen sets at each point of X as follows: Each $\{\alpha, \beta\} \in F_2(G) - - F_1(G)$ is isolated. If $(\alpha, f) \in \omega_2 \times 2^\omega$ and $n < \omega$, set $B_n(\alpha, f) = \{(\alpha, g): g \restriction n = f \restriction n\} \cup \{\{\alpha, \gamma\}: \alpha — \gamma,\ h(\gamma) \restriction n = = f \restriction n \text{ and } h(\gamma) \neq f\}$. X is first countable, 0-dimensional, T_2 and locally compact - each $B_0(\alpha, f)$ is "similar" to a closed subspace of the Alexandrov double of 2^ω. For each $\alpha < \omega_2$, set $V_\alpha = [\{\alpha\} \times 2^\omega] \cup [\{\{\alpha, \gamma\}: \alpha — \gamma\}]$. Each V_α is a compact open set of X and hence is clopen in any compactification of X. Since $V_\alpha \cap V_\beta \neq \phi$ iff α —— β , we see that no compactification of X is a remainder.

Let us call a space X <u>σ-linked</u> if the topology of X is the union of countably many linked collections.

<u>Problem 4.4</u>. Is a σ-linked compact T_2 space a remainder of ω ?

No counterexample could be supercompact since E. van Douwen [1] has proven that all supercompact σ-linked spaces are separable. A possible counterexample is the Stone space of the Lebesgue measurable subsets of [0,1] modulo the ideal of null sets.

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