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ON A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract: Ω is a bounded domain with smooth boundary $\partial\Omega$ and L is a linear properly elliptic partial differential operator (not necessarily self-adjoint) of order m with smooth real coefficients on $\bar{\Omega}$. $\{B_j\}$, $1 \leq j \leq \frac{1}{2}m$ is a set of $\frac{1}{2}m$ differential boundary operators which cover L and have smooth coefficients on $\partial\Omega$. A is L acting on functions satisfying the boundary conditions:

$B_j u = 0$ on $\partial\Omega$, $1 \leq j \leq \frac{1}{2}m$. $g: \Omega \times R \to R$ is a function. The purpose of this paper is to seek a solution of $A(u) = g(x,u)$ under conditions different from the known ones. It is assumed that 0 is an eigenvalue of A.

Key words: Elliptic operator, boundary value problem.

Classification: Primary 47H15
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1. Introduction. Let Ω be a bounded domain with smooth boundary $\partial\Omega$ and L be a linear properly elliptic partial differential operator of order m with smooth real valued coefficients on $\bar{\Omega}$. Let $\{B_j\}$ be a set of $\frac{1}{2}m$ differential boundary operators with real valued coefficients smooth on $\partial\Omega$ which covers L (for definitions and further descriptions of such problems see [5] and [8]). Let A be the operator L acting on functions which satisfy the boundary conditions:

$$B_j u = 0 \text{ on } \partial\Omega, \ 1 \leq j \leq \tfrac{1}{2}m \tag{1.1}$$

The operator A when considered as defined on $L^2(\Omega)$ is closable. We may, therefore, regard A as a closed operator with domain $A\subset L^2(\Omega)$. It is known that A is a Fredholm operator, i.e., $R(A)$, the range space of A is closed in $L^2(\Omega)$ and both $R(A)^{\perp}$ and the null space $N(A)$ are finite dimensional (see [8]). Throughout the paper we will assume that $N(A)\neq\{0\}$. Let $g:\Omega\times R\to R$ be a function such that for each $t\in R$ the function $x\to g(x,t)$ is measurable in Ω and for each $x\in\Omega$ (a.e.) the function $t\to g(x,t)$ is continuous in R. Assume that there exists a function $\tilde{g}(x)\in L^1(\Omega)$ such that

(1.2) $$|g(x,t)|\leq\tilde{g}(x),\quad x\in\Omega\quad, t\in R$$

Further assume that there exist functions $g_{\pm}(x)\in L^1(\Omega)$ such that

(1.3) $$\lim_{t\to\pm\infty} g(x,t) = g_{\pm}(x)\quad \text{a.e.}$$

Let $T:R(A)^{\perp}\longrightarrow N(A)$ be a linear mapping such that

(1.4) $$\int_{T(z)>0} g_{+}(x)z(x)dx + \int_{T(z)<0} g_{-}(x)z(x)dx > 0$$

for each nonzero $z\in R(A)^{\perp}$, where $T(z)\gtrless 0 = \{x\in\Omega:(Tz)(x)\gtrless 0$.

Also let

(1.5) $$\wp_T(c)\longrightarrow 0 \text{ as } c\longrightarrow 0$$

where

$$\wp_T(c) = \sup_{z\in R(A)^{\perp}} \text{measure}\,\{x\in\Omega: z(x)\neq 0,\ \frac{|(T(z))(x)|}{|z(x)|}<c\}.$$

Under the conditions (1.1) to (1.5) Schechter [9] has proved that there exists $u\in \text{dom}\,A$ such that

$$A(u) = g(x,u)$$

This type of problem has been considered first by Landesman and Lazer [4] and then by Williams [10], Browder [1], Nirenberg ([6], [7]), Schechter [9] and many others. In [4] Landesman and Lazer

have considered the Dirichlet problem where the above operator A is self-adjoint second order with dimension of $N(A) = 1$ and $g(x,t)$ is of the form $h(x) - g(t)$ and is continuous.
Assuming that $\lim_{t \to \pm\infty} g(x,t) = g_{\pm}(x) \in L^2(\Omega)$, $N(A)$ is spanned by w with $\|w\|_{L^2(\Omega)} = 1$ and a condition corresponding to (1.2), Landesman and Lazer ([4], Theorem 5.2) have proved that there exists $u \in D(A)$ satisfying $A(u) = g(x,u)$ if

$$(1.6)\ \left[\int_{w>0} g_+(x)w(x)dx + \int_{w<0} g_-(x)w(x)dx\right]\left[\int_{w>0} g_-(x)w(x)dx + \int_{w<0} g_+(x)w(x)dx\right] < 0$$

This result of Landesman and Lazer [4] has been extended by Williams [10] to the case where A is higher order self-adjoint operator and $N(A)$ is of arbitrary finite dimension and by Browder [1] to the case where A is arbitrary self-adjoint operator and $N(A)$ is of arbitrary finite dimension. Nirenberg [7] is the first to deal with the case when A is non self-adjoint and $N(A)$ is of arbitrary finite dimension. The result of Nirenberg [7] involves assumptions expressed in terms of nonvanishing of degree of a certain map when $\mathrm{Ind}\, A = \dim N(A) - \dim R(A)^{\perp} = 0$ and the nontriviality of the stable homotopy class of a certain map when $\mathrm{Ind}\, A > 0$, while that of Schechter [9] mentioned at the beginning involves conditions (1.4) expressed in terms of inequalities which are easy to verify.

Since all these results have grown out of the paper of Landesman and Lazer [4] it is of considerable interest to see if the result of Schechter can be proved with condition analogous to condition (1.6) of Landesman and Lazer, i.e. if condition (1.4) can be replaced by

$$(1.7)\ \Big[\int_{T(z)>0} g_+(x)z(x)dx + \int_{T(z)<0} g_-(x)z(x)dx\Big]\Big[\int_{T(z)>0} g_-(x)z(x)dx + \int_{T(z)<0} g_+(x)z(x)dx\Big] < 0$$

for each nonzero $z \in R(A)^\perp$.

In this paper a little more than this has been achieved. The condition (1.7) is indeed analogous to the condition (1.6), for if A is self-adjoint, $R(A)^\perp = N(A)$ and T can therefore be taken as the identity operator. We also note that in this case the condition (1.5) is automatically satisfied.

Our approach is via a simple theorem of Krasnosel'skii [3] on degree theory, the application of which seems to the best of the author's knowledge to be new.

2. A fixed point theorem. In this section we will prove a fixed point theorem for which we need the following result, due to Krasnosel'skii [3], which we write as a lemma.

Lemma 2.1. Let X be a real Banach space and $D \subset X$ an open bounded set symmetric with respect to the origin and containing it.

Let $T:\bar{D} \to X$ be a compact mapping (i.e. T is continuous and $T(\bar{D})$ is relatively compact in X) such that

$(I - T)(x) \neq \mu(I-T)(-x)$ for every $\mu \in [0,1]$ and every $x \in \partial D$,

the boundary of D, where I is the identity on X. Then there exists at least one $x \in D$ such that $T(x) = x$.

Theorem 2.1. Let X be a real Banach space and Z a finite dimensional real Hilbert space. Let $T:Z \to X$, $G:X \to X$ and H:

$:X \longrightarrow Z$ be all compact mappings (i.e. mappings which are continuous and map a bounded set onto a relatively compact set).

Assume that

(i) $\limsup_{\|u\| \to \infty} \frac{\|G(u)\|}{\|u\|} = \tilde{\beta} < 1$

and (ii) for large $\|z\|$,

$(z, H(T(z) + G(u)) - \mu H(T(-z) + G(-u))) \neq 0$

for all u and $\mu \in [0,1]$ where $(.,.)$ denotes the inner product in Z.

Then the mapping $\hat{T}: X \times Z \to X \quad Z$ defined by

$\hat{T}((u,z)) = (u^*, z^*) = (T(z) + G(u),\ z - H(T(z) + G(u)))$

has a fixed point.

__Proof.__ Clearly, $X \times Z$ is a real Banach space with the norm $\|(u,z)\| = \|u\| + \|z\|$, $u \in X$, $z \in Z$ and $\hat{T}$ is a compact mapping on $X \times Z$.

By virtue of Lemma 2.1 it would suffice to show that there is $D \subset X \times Z$ a bound open set containing the origin and symmetric with respect to the origin such that

$$(2.1) \qquad (I - \hat{T})(u,z) \neq \mu(I - \hat{T})(-u,-z)$$

for every $\mu \in [0,1]$ and every $(u,z) \in \partial D$, I being the identity on $X \times Z$.

Let R' be a positive real number such that condition (ii) holds for $\|z\| = R > R'$. By assumption (i) there exists β with $0 \leq \beta < 1$ and $R \geq R'$ such that

$$\|G(u)\| \leq \beta \|u\| \text{ for } \|u\| > R.$$

There are also constants K_1 and K_2 such that

$$\|T(z)\| \leq K_1 \text{ whenever } \|z\| \leq R$$

and $$\|G(z)\| \leq K_2 \text{ whenever } \|u\| \leq R.$$

Now let for some $(u,z) \in X \times Z$ and some $\mu \in [0,1]$

$$(I - \hat{T})(u,z) = \mu(I - \hat{T})(-u,-z),$$

i.e. $(u,z) - (u^*,z^*) = \mu(-u,-z) - \mu((-u)^*, (-z)^*)$

which yields

$$(2.2) \quad (1 + \mu)u = u^* - \mu(-u)^* = T(z) + G(u) - \mu(T(-z) + G(-u))$$

and $(1 + \mu)z = z^* - \mu(-z)^* = z - H(T(z) + G(u)) - \mu(-z - H(T(-z) + G(-u))$, i.e. $H(T(z) + G(u)) - \mu H(T(-z) + G(-u)) = 0$, which in view of condition (ii) implies that $\|z\| \leq R$. Let $M = \max\left(\frac{K_1}{1-\beta}, K_1 + K_2\right)$.

Now if $\|u\| > R$ we have from (2.2)

$$(1 + \mu)\|u\| \leq \|T(z)\| + \|G(u)\| + \mu(\|T(-z)\| + \|G(-u)\|) \leq K_1 + \beta\|u\| + \mu K_1 + \mu\beta\|u\| \quad \text{as} \quad \|z\| \leq R.$$

Thus $\|u\| \leq \frac{K_1}{1-\beta} \leq M$.

When $\|u\| \leq R$ we have again from (2.2)

$$(1 + \mu)\|u\| \leq K_1 + K_2 + \mu(K_1 + K_2), \text{ i.e. } \|u\| \leq K_1 + K_2 \leq M.$$

Thus in either case $\|u\| \leq M$. The constants R and M are independent of μ. Let $\hat{R}$ be any real number greater than $R + M$ and $D = \{(u,z) \in X \times Z : \|(u,z)\| < \hat{R}\}$. Clearly then (2.1) holds with this D and the proof is complete.

3. __Main results__. In this section we prove the existence of the solution of the nonlinear boundary value problem $A(u) = g(x,u)$.

__Theorem 3.1__. Let Ω, L, B_j $(1 \leq j \leq \frac{1}{2}m)$ and A be as in the beginning of section 1. Also let $g: \Omega \times R \to R$, $\gamma(x) \in L^1(\Omega)$

satisfying (1.2) and $g_{\pm}(x) \in L^1(\Omega)$ satisfying (1.3) be as in section 1.

Noting that $A:\mathrm{dom}\, A \subset L^2(\Omega) \to L^2(\Omega)$ is a Fredholm mapping and $N(A) \neq \{0\}$ by assumption (vide section 1) let $T:R(A)^{\perp} \to$ $\to N(A)$ be a linear mapping. Assume that

(a) for each $0 \neq z \in R(A)^{\perp}$ and each $\mu \in [0,1]$,

$$(3.1)\quad M_T(z,\mu) = \int_{T(z)>0} g_+(x)z(x)dx + \int_{T(z)<0} g_-(x)z(x)dx -$$

$$- \mu\Big[\int_{T(z)>0} g_-(x)z(x)dx + \int_{T(z)<0} g_+(x)z(x)dx\Big] \neq 0$$

(3.2) (b) $\qquad \rho_T(c) \to 0$ as $c \to 0$

where $\rho_T(c)$ is as defined in section 1.

Then there exists $u \in \mathrm{dom}\, A$ such that $A(u) = g(x,u)$.

Before proving this theorem we note that (3.1) implies that ind $A \geqq 0$ for T satisfying (3.1) is injective. To see this let $z_0 \in N(T)$. Then $M_T(z_0,\mu) = 0$ for every $\mu \in [0,1]$ contradicting (3.1).

<u>Proof</u>. We will maintain the notation and follow more or less the same argument of [9]. Let us assume that $\dim R(A)^{\perp} =$ $= n$ and $(z_1, z_2, \ldots, z_n)$ be an orthonormal basis for $R(A)^{\perp}$. Let P be the projection of $L^2(\Omega)$ onto $R(A)^{\perp}$ defined by

$$P(h) = \sum_{k=1}^{n} (h, z_k), h \in L^1(\Omega).$$

It follows that P maps $L^1(\Omega)$ into $L^{\infty}(\Omega)$, z_k, $k = 1,2,\ldots,n$ being smooth in $\bar{\Omega}$. From the linear theory of elliptic boundary value problems it is known that there is a linear operator $S:R(A) \to N(A)^{\perp}$ such that S is the inverse of A restricted to $N(A)^{\perp}$, $S(I - P)$ maps $L^1(\Omega)$ into $L^p(\Omega)$ for some $p > 1$

and is compact (for details see [8] and [9]). We will apply Theorem 2.1 and to this end we take $X = L^p(\Omega)$, $p > 1$ obtained as above, $Z = R(A)^{\perp}$ and define $G: X \longrightarrow X$ and $H: X \longrightarrow Z$ by

$$G(u) = S(I - P)g(x,u), \quad u \in X$$

and

$$H(u) = Pg(x,u), \quad u \in X.$$

Obviously T, G and H are all compact mappings. It can be proved (see Schechter [9]) that

$$\tilde{\beta} = \lim_{\|u\| \to \infty} \sup \frac{\|G(u)\|}{\|u\|} = 0$$

and that there is a constant σ such that

$$\|G(u)\| \leq \sigma \quad \text{for all } u \in X.$$

We now verify the condition (ii) of Theorem 2.1. Let $u \in X$, $0 \neq z \in Z$, $\mu \in [0,1]$ and $\varepsilon > 0$ be given. Since by assumption $\tilde{g} \in L^1(\Omega)$, there exists $\delta > 0$ such that

(3.3) $$\int_W \tilde{g}(x)dx < \varepsilon/_{24}$$

for any $W \subset \Omega$ with $m(W) < \delta$ where $m(A)$ denotes the measure of $A \subset \Omega$. Let $W_1 = \{x \in \Omega : |G(u)(x)| > \frac{3\sigma}{\delta}\}$. Then $m(W_1) < \frac{\delta}{3}$. Again by (3.2) there is a positive integer N independent of z such that $m(W_2) < \frac{\delta}{3}$ where $W_2 = \{x \in \Omega : \frac{|(T(z))(x)|}{|z(x)|} < \frac{1}{N}\}$. Also by (1.3) and Egoroff Theorem there is a set $W_3 \subset \Omega$ with $m(W_3) < \frac{\delta}{3}$ and a positive constant J such that

(3.4) $$|g(x,t) - g_{\pm}(x)| < \varepsilon/_{12m(\Omega)}$$

holds for $\pm t > J$ and $x \in \Omega \setminus W_3$. Let $L = \frac{24}{\varepsilon} \int_\Omega \tilde{g}(x)dx$ and set $W = \bigcup_{i=1}^{3} W_i$. Clearly $m(W) < \delta$.

Lastly let $D = \{x \in \Omega \setminus W : |(T(z))(x)| < \|z\|_\infty / LN$ and $E = \Omega \setminus (D \cup W)$.

We now consider the following:

$$\left| \int_\Omega [g(x,u^*) - \mu g(x,(-u)^*)]\, z(x)dx - M_T(z,u) \right|$$

$$\leq \int_{T(z)>0} \left| [\{g(x,u^*) - g_+(x)\} - \mu\{g(x,(-u)^*) - g_-(x)\}]\, z(x) \right| dx + \int_{T(z)<0} \left| [\{g(x,u^*) - g_-(x)\} - \mu\{g(x,(-u)^*) - g_+(x)\}]\, z(x) \right| dx = I_1 + I_2,$$

where $(u)^*$ has been defined in Theorem 2.1.

Now $I_1 = \int_{T(z)>0} \left| [\{g(x,u^*) - g_+(x)\} - \mu\{g(x,(-u)^*) - g_-(x)\}]\, z(x) \right| dx \leq \int_{W\cap[T(z)>0]} + \int_{D\cap[T(z)>0]} + \int_{E\cap[T(z)>0]}$

By (3.3) we have

$$\int_{W\cap[T(z)>0]} \leq 2(1+\mu)\, \|z\|_\infty \int_{W\cap[T(z)>0]} \tilde{g}(x)dx) \leq 4\, \|z\|_\infty \frac{\varepsilon}{24} = \frac{\varepsilon}{6} \|z\|_\infty .$$

Also $\int_{D\cap[T(z)>0]} \leq \frac{2(1+\mu)}{L} \|z\|_\infty \int_\Omega \tilde{g}(x)dx \leq \frac{4}{L} \|z\|_\infty \int_\Omega \tilde{g}(x)dx$

$\leq \frac{\varepsilon}{6} \|z\|_\infty$ as $|z(x)| < \|z\|_\infty / L$ on D.

We now take $\|z\|_\infty \geq L\,N(J + \frac{3\sigma}{\sigma})$, the right hand side being independent of z, u and μ .

Now on $E\cap[T(z)>0]$

$$T(z) + G(u) \geq T(z) - |G(u)| \geq \frac{\|z\|_\infty}{LN} - \frac{3\sigma}{\sigma} \geq J$$

and $T(-z) + G(-u) \leq -T(z) + |G(-u)| \leq \frac{-\|z\|_\infty}{LN} + \frac{3\sigma}{\sigma} \leq -J.$

Hence by (3.4) we have

$$\int_{E\cap[T(z)>0]} \leq \left| \frac{\varepsilon}{12m(\Omega)} + \mu \frac{\varepsilon}{12m(\Omega)} \right| \|z\|_\infty\, m(\Omega) = \frac{\varepsilon}{6} \|z\|_\infty .$$

Thus we have proved that $I_1 < \frac{\varepsilon}{2} \|z\|_\infty$.

Proceeding exactly as above and noting only that on $E\cap(T(z)<$

$$T(z) + G(u) \leqq T(z) + |G(u)| \leqq \frac{-\|z\|_\infty}{LN} + \frac{3\sigma}{\sigma'} \leqq -J$$

and $$T(-z) + G(-u) \geqq -T(z) - |G(-u)| \geqq \frac{\|z\|_\infty}{LN} - \frac{3\sigma}{\sigma'} \geqq J$$

we can show that $I_2 \leqq \frac{\varepsilon}{2} \|z\|_\infty$.

Hence, replacing z by tz, $t > 0$, we obtain

$$(3.5) \quad V(t,z,u,\mu) = |(g(x,t(T(z)) + G(u)) - \mu g(x,-t(T(z)) + G(-u)),z) - M_T(z,u)| \leqq \varepsilon \|z\|_\infty$$

whenever $t \geqq LN(j + \frac{3\sigma}{\sigma'})/\|z\|_\infty = K(\varepsilon)/\|z\|_\infty$ and hence the left hand side of (3.5) tends to zero uniformly in u, μ and z provided $\|z\|$ and $1/\|z\|$ is bounded. $V(t,z,u,\mu)$ being continuous in the variable (z,μ) for each t and closed bounded sets in $Z \times [0,1]$ being compact, it follows that $M_T(z,\mu)$ is continuous in (z,μ) for $z \neq 0$. Now since the set $A = \{(z,\mu): \|z\| = 1$ and $\mu \in [0,1]\}$ is compact and connected in $Z \times [0,1]$, it follows that $M_T(A)$ is a closed bounded interval $[a,b]$, say. Again by virtue of (3.1), $0 \notin [a,b]$. Hence either (1) $[a,b]$ consists only of negative real numbers or (2) $[a,b]$ consists only of positive real numbers. In case (1) we have $M_T(z,u) < \frac{b}{2}\|z\|$ for all $0 \neq z \in Z$ and all $\mu \in [0,1]$. Using $\varepsilon = \frac{|b|}{2}$ in (3.5) we obtain that for sufficiently large $\|z\|$,

$$g(x,T(z) + G(u)) - \mu g(x,-T(z) + G(-u)),z) \neq 0$$

for all u and $\mu \in [0,1]$.

In case (2) we have $M_T(z,\mu) > \frac{a}{2}\|z\|$ for all $0 \neq z \in Z$ and all $\mu \in [0,1]$. Using $\varepsilon = \frac{a}{2}$ in (3.5), we obtain that for sufficiently large $\|z\|$

$$(g(x,T(z) + G(u)) - \mu g(x,-T(z) + G(-u)),z) \neq 0$$

for all u and $\mu \in [0,1]$.

Thus in either case for sufficiently large $\|z\|$

$$(3.6)\quad (g(x,T(z)+G(u)) - \mu g(x,-T(z)+G(-u)),z) \neq 0$$

for all u and $\mu \in [0,1]$.

Now let $z = \sum_{i=1} \alpha_i z_i$. Then

$$\begin{aligned}
&(H(T(z)+G(u)) - \mu H(T(-z)+G(-u)),z)\\
(3.7)\quad &= (\sum_{i=1}^{m} (g(x,T(z)+G(u) - \mu g(x,T(-z)+G(-u)),z_i)z_i,\\
&\qquad \sum_{i=1}^{m} \alpha_i z_i) = (g(x,T(z)+G(u)) - \mu g(x,T(-z)+\\
&\quad + G(-u)),z) \neq 0
\end{aligned}$$

for all u and $\mu \in [0,1]$ and for sufficiently large $\|z\|$.

Thus the condition (ii) of Theorem 2.1 is verified.

It is trivial to see that if (u,z) is the fixed point obtained by Theorem 2.1, then $u \in D(A)$ and $A(u) = g(x,u)$. Thus the proof is complete.

<u>Corollary 3.1.</u> Let Ω, L, B_j $(1 \leq j \leq \frac{1}{2}m)$, A, T, g, $\tilde{g}$, $g_\pm$ be as in Theorem 3.1. Let the condition (b) of Theorem 3.1 hold. Further assume that the following holds:

for each $0 \neq z \in R(A)^\perp$

$$\begin{aligned}
(3.8)\quad &\Big[\int_{T(z)>0} g_+(x)z(x)dx + \int_{T(z)<0} g_-(x)z(x)dx\Big]\\
&\Big[\int_{T(z)>0} g_-(x)z(x)dx + \int_{T(z)<0} g_+(x)z(x)dx\Big] < 0
\end{aligned}$$

Then there exists $u \in \operatorname{dom} A$ such that $A(u) = g(x,u)$.

<u>Proof.</u> The condition (3.8) implies the condition (a) of Theorem 3.1 and hence the corollary is proved.

<u>Corollary 3.2.</u> Let Ω, L, B_j $(1 \leq j \leq \frac{1}{2}m)$, A, T, g, $\tilde{g}$, $g_\pm$ be as in Theorem 3.1. Let the condition (b) of Theorem 3.1 hold. Furthermore let either of the following conditions hold:

(i) for each $0 \neq z \in R(A)^{\perp}$

$$\left[\int_{T(z)>0} g_+(x)z(x)dx + \int_{T(z)<0} g_-(x)z(x)dx\right] > 0;$$

(ii) for each $0 \neq z \in R(A)^{\perp}$

$$\left[\int_{T(z)>0} g_+(x)z(x)dx + \int_{T(z)<0} g_-(x)z(x)dx\right] < 0.$$

Then there exists $u \in \operatorname{dom} A$ such that $A(u) = g(x,u)$.

<u>Proof</u>. Setting $Q(z) = \int_{T(z)>0} g_+(x)z(x)dx + \int_{T(z)<0} g_-(x)z(x)dx$ and noting that T is linear, it follows that for any $z \in R(A)^{\perp}$

$$-Q(-z) = \int_{T(z)>0} g_-(x)z(x)dx + \int_{T(z)<0} g_+(x)z(x)dx.$$

Let us now assume that the condition (i) holds. Then for each $0 \neq z \in R(A)^{\perp}$, $Q(z) > 0$ and $Q(-z) > 0$ and hence $Q(z)[-Q(-z)] < 0$ which is the condition (3.8) of Corollary 3.1. Similarly we can prove the corollary under condition (ii).

<u>Remark 3.1</u>. The corollary 3.1 includes the result of Schechter [9]. We should also point out that the condition (3.8) of Corollary 3.1 or more generally the condition (3.1) of Theorem 3.1 implies that either condition (i) or condition (ii) of Corollary 3.2 holds. This follows from the continuity of $M_T(z, \mu)$ asserted in the proof of Theorem 3.1 and the fact that the set $\{(z, \mu): \|z\| = 1$ and $\mu \in [0,1]\}$ is closed and compact (see Theorem 3.1). Thus under condition 3.1 the possible new hypotheses are limited to either (i) or (ii).

<u>Remarks 3.2</u>

1. The condition of (b) of Theorem 3.1 holds if A has the unique continuation property, i.e. the only solution of $A(u) = 0$ which vanishes on a set of positive measure in Ω is $u = 0$

(for proof of this result see Lemma 2 in [7] or [2], p. 160).

2. Nirenberg's remark in [9] that instead of assuming T to be linear, it is sufficient to assume T to be continuous and homogeneous and ind A to be $\geqq 0$ is also valid in our case.

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