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ON THE REPRESENTATION OF ORTHOCOMPLEMENTED POSETS

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Abstract: The possibility of the representation of orthomodular orthoposets is discussed in [3],[4],[7] Klukowski [4] used the notion of ultrafilter, which has been introduced by O. Frink [2], for any poset and proved the theorem of Stonean type for Boolean weakly orthomodular orthoposets. In this paper the notion of an M-base defined by A.R. Marlow [5] is used as a convenient tool for the construction of the representation of orthocomplemented poset. Some consequences of the representation theorem are deduced.

Key words: Poset, Boolean algebra, ultrafilter of Boolean algebra, Stone space and related topological notions.

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§ 1. Basic notions and definitions

Definition 1 [3]. An *orthocomplemented poset* is a partially ordered set $(P, \leq, 0, 1, ')$ containing a universal lower bound 0, a universal upper bound 1, and having a unary operation $':P \longrightarrow P$ called *orthocomplementation* which for any $a,b \in P$ satisfies

(i) $a \leq b$ implies $b' \leq a'$

(ii) $(a')' = a$ for each $a \in P$

(iii) $a \wedge a' = 0$ and $a \vee a' = 1$, $a \in P$.

The elements $a,b \in P$ are said to be orthogonal if $a \leq b'$. We shall write then $a \perp b$. In a contrary case, i.e. if $a \nleq b'$ for $a,b \in P$, we shall call a, b mutually non-orthogonal, and then we write $a \not\perp b$.

Definition 2. Let $(P, \leq, 0, 1, ')$ be an orthocomplemented poset. A nonempty subset $\emptyset \neq M \subset P$ is said to be an N-set of P, if for any $a, b \in M$ $a \not\perp b$ holds. The N-set $M_0 \subset P$ is a maximal N-set, if there is no such N-set $M \subset P$ that $M_0 \subset M$, $M_0 \neq M$.

Proposition 1. If $(P, \leq, 0, 1, ')$ is an orthocomplemented poset, $p \in P$, $p \neq 0$, then there exists such a maximal N-set $M \subset P$, that $p \in M$.

Proof: It is obvious that $A = \{p\}$ is an N-set. Let X be the set of all N-sets of P containing the element p. X is partially ordered by inclusion. Let $\{M_\alpha\}_{\alpha \in S}$ (S - the set of indexes) be a chain in X. The set $D = \bigcup_{\alpha \in S} M_\alpha$ is also an N-set. The validity of the proposition is then a consequence of Zorn's lemma.

Definition 3. Let $(P_1, \leq, 0_1, 1_1, ')$, $(P_2, \preceq, 0_2, 1_2, *)$ be two orthocomplemented posets. A mapping $f: P_1 \longrightarrow P_2$ is called an orthomorphism, if

(i) $a, b \in P_1$, $a \leq b$ implies $f(a) \preceq f(b)$

(ii) $f(a') = [f(a)]^*$ for each $a \in P_1$

(iii) $f(0_1) = 0_2$

An orthomorphism $f: P_1 \longrightarrow P_2$ which is bijective, and such that the inverse mapping $f^{-1}: P_2 \longrightarrow P_1$ is also an orthomorphism is said to be an orthoisomorphism. We shall call then the posets P_1, P_2 orthoisomorphic.

§ 2. M-bases and their characterization. The notion of M-base was introduced by A.R. Marlow [5] for logics. Without any modification we can use the definition of M-base also for orthocomplemented posets.

__Definition 4__ [5]. Let $(P, \leq, 0, 1, ')$ be an orthocomplemented poset. The non-empty subset $\emptyset \neq B \subset P$ is called an M-base of P, if

(i) $1 \in B$

(ii) $\{p, p'\} \cap B \neq \emptyset$ for each $p \in P$

(iii) If $p \in P$, $q \in B$, $q \perp p$ then $p \notin B$.

__Lemma 1.__ Let $(P, \leq, 0, 1, ')$ be an orthocomplemented poset, then the following conditions are equivalent:

(a) The set $B \subset P$ is an M-base of P

(b) The set $B \subset P$ satisfies the conditions

I $p \in B$, $p \leq q$ implies $q \in B$

II $\mathrm{card}\, [\{p, p'\} \cap B] = 1$ for each $p \in P$

(c) B is a maximal N-set.

Proof: (a) $\Longrightarrow$ (b)

(b) I Let $p \in B$, $q \in P$ and $p \leq q$. Since $p \leq q = (q')'$, we get $p \perp q'$. Now (ii), (iii) of Definition 4 implies $q' \notin B$. Therefore $q \in B$.

(b) II follows immediately from (ii) and (iii) of Definition 4.

(b) $\Longrightarrow$ (c). Assume that the set $B \subset P$ satisfies (b)I, (b)II, and let $p, q \in B$. Then $p \not\perp q$. Indeed, if $p \leq q'$ then (b)I would imply $q' \in B$, which contradicts (b)II. We prove that B is a maximal N-set.

Let B_1 be such an N-set in P that $B \subset B_1$, $B \neq B_1$. If $p \in B_1 \setminus B$, then by (b)II we should have $p' \in B \subset B_1$. But this last argument contradicts the fact B_1 being an N-set. The validity of (c) is now established.

(e) $\Longrightarrow$ (a). Let $B \subset P$ be a maximal N-set. We shall show that B satisfies (i) - (iii) of Definition 4.

(i) For each $p \in B$, $p \neq 0$ we have $1' = 0 < p$. Therefore $p \not\perp 1$. The maximality of the N-set P implies $1 \in B$.

(ii) Let $p \in P$, and assume that $p \notin B$, $p' \notin B$. Maximality of the N-set B implies the existence of such elements $q_1, q_2 \in B$ that $p \perp q_1$ and $p' \perp q_2$. From this it follows $q_1 \perp q_2$ - a contradiction. Now it can be easily seen that for each $p \in P$, card $[\{p,p'\} \cap B] = 1$.

(iii) Let $p \in P$, $q \in B$ and $q \perp p$. Then $p \notin B$ because B is an N-set.

__Corollary__. Let $(P, \leq, 0, 1, ')$ be an orthocomplemented poset. If $p \in P$, $p \neq 0$, then always such an M-base B exists in P, that $p \in B$.

Proof: evident.

Remember that if $(P, \leq)$ is a poset, $p, q \in P$, $p \leq q$, then $\langle p,q \rangle = \{x \in P \mid p \leq x \leq q\}$.

The following lemma shows a method how to construct new M-bases from a given one.

__Lemma 2__. Let $(P. \leq, 0, 1, ')$ be an orthocomplemented poset, B_0 an M-base of P, $p \in P \setminus B_0$, $p \neq 0$. Then the set $B_1 = (B \setminus \langle 0,p' \rangle) \cup \langle p,1 \rangle$ is an M-base containing p.

Proof: Follows immediately. It suffices to verify the validity of conditions (b)I, (b)II of Lemma 1 for B_1.

__Corollary__. If $(P, \leq, 0, 1, ')$ is a Boolean algebra, then each ultrafilter of P is an M-base in P.

Proof: evident.

But the contrary assertion may be false.

Proposition 2. Let $(P, \leq, 0, 1, ')$ be such a Boolean algebra that card $B \geq 8$. Then P contains an M-base, which is not an ultrafilter.

Proof: Let B_1 be any M-base in P. First of all we shall show that we can always find such elements $p, q \in B_1$ for which $p \nleq q$, $q \nleq p$. Suppose, on the contrary, that for every $p, q \in B_1$ holds either $p \leq q$ or $q \leq p$. Because card $B_1 \geq 4$ must such $p_i \in B_1$, $i = 1,2,3$ exist that $p_1 < p_2 < p_3 < 1$. Now let us take the element $a = p_3 \wedge p_2'$. Then $a \leq p_2'$, $p_2' \notin B_1$ and it follows that $a \notin B_1$. Therefore $a' \in B_1$. But $a' = (p_3 \wedge p_2')' = p_3' \vee p_2$. The fact that neither $p_3' \vee p_2 \leq p_3$ nor $p_3 \leq p_3' \vee p_2$ contradicts the assumption about B_1.

Now let B_0 be an ultrafilter in P. By Corollary of Lemma 2 B_0 is an M-base. Let further p, q be such elements of B_0 that $p \nleq q$, $q \nleq p$. Then $p \wedge q \neq 0$, $p \wedge q \in B_0$ because B_0 is a proper filter. Lemma 2 implies that $B_1 = (B_0 \setminus \langle 0, p \wedge q \rangle) \cup \langle (p \wedge q)', 1 \rangle$ is an M-base in P. But $p \wedge q \notin B_1$ although $p, q \in B_1$.
Therefore B_1 is not an ultrafilter in P. This completes the proof.

Remark. With little modifications one can prove an analogical proposition for the so-called Boolean orthomodular orthoposets. In this case the ultrafilters are considered in the sense of Frink's definition [2].

§ 3. Representation theorem for orthocomplemented posets

Notations. Let $(P, \leq, 0, 1, ')$ be an orthocomplemented poset and denote by $M(P)$ the set of all M-bases in P. If $p \in P$, $p \neq 0$ put $Z(p) = \{B \in M(P) \mid B \ni p\}$ and let $Z(0) = \emptyset$. Finally we put

$Z(M(P)) = \{Z(p) \mid p \in P\}$. Then the following theorem of the Stonean type turns out to be valid.

Theorem 1. Every orthocomplemented poset $(P, \leq, 0, 1, ')$ is orthoisomorphic with the orthocomplemented poset $(Z(M(P)), \subseteq, \emptyset, M(P), {}^*)$ the elements of which, the sets $Z(p)$, $p \in P$ are clopen subsets of zero-dimensional completely regular topological T_1-space $X = (M(P), \mathcal{T})$. The set $Z(M(P))$ is a subbasis for the topology $\mathcal{T}$. The symbols $\subseteq$ and $*$ denote the inclusion relation and set-theoretical complement in $M(P)$ respectively.

Proof: $M(P) \neq 0$ by Corollary of Lemma 1. Now we introduce a topology $\mathcal{T}$ on $M(P)$, requiring that $Z(M(P))$ be a subbasis for closed subsets of $M(P)$.

(i) The class $Z(M(P))$ is also a subbasis for open sets of the topological space $(M(P), \mathcal{T})$. Indeed, if $B \in M(P)$, then there exists such $p_0 \in P$, $0 \neq p_0 \neq 1$ that $p_0 \in B$. Therefore $B \in Z(p_0)$. Now bII of Lemma 1 implies $Z(p) = M(P) \setminus Z(p')$ for each $p \in P$. Therefore the sets $Z(p)$ are open and it is also clear that $Z(M(P))$ is a subbasis for open sets in $(M(P), \mathcal{T})$.

(ii) $\mathcal{T}$ is a Hausdorff topology on $M(P)$. Let $B_1, B_2 \in M(P)$, $B_1 \neq B_2$. Then there exists such $p \in P$ that $p \in B_1$, $p' \in B_2$. The open sets $Z(p)$, $Z(p')$ are then disjoint neighbourhoods of B_1, B_2 respectively.

(iii) The topological space $(M(P), \mathcal{T})$ is zero-dimensional. In fact, the basis $\mathcal{U}$ of open sets of the topology $\mathcal{T}$ is of the form $\mathcal{U} = \{U \subset M(P) \mid U = \bigcap_{i=1}^{n} Z(p_i),\ p_i \in P,\ i = 1,2,\dots,n\}$. Since $Z(p_i)$ are clopen sets, it follows that the sets $U \in \mathcal{U}$ are also clopen.

(iv) The topological space $(M(P), \mathcal{T})$ is completely regular. This is a simple consequence of (ii) and (iii).

(v) $(Z(M(P)), \subseteq, \emptyset, M(P), {}^*)$ is an orthocomplemented poset. The set $Z(M(P))$ is partially ordered by the inclusion relation $\subseteq$. If $A \in Z(M(P))$, then we put $A^* = M(P) \setminus A$. Clearly $Z(1) = M(P)$, $Z(0) = \emptyset$, and $M(P)$ and $\emptyset$ are the universal upper and lower bounds in $Z(M(P))$ respectively. According to the relation $Z(p') = M(P) \setminus Z(p)$, $p \in P$ we obtain

(1) $[Z(p)]^* = M(P) \setminus Z(p) = Z(p')$ for each $p \in P$.

It can be easily seen that $*$ satisfies all requirements imposed on orthocomplementation.

(vi) If $p, q \in P$, then $p \leq q \iff Z(p) \subseteq Z(q)$.

(a) Let $p \leq q$. The property (b)I of Lemma 1 implies $Z(p) \subseteq Z(q)$.

(b) Assume $Z(p) \subset Z(q)$. If $p = 0$, then clearly $0 = p \leq q$. Also let $p \neq 0$, and suppose that $p \nleq q$. Then we can select such an M-base B that $B \in Z(p)$. Following Lemma 2 $B_1 = (B \setminus \langle 0, q \rangle) \cup \langle q', 1 \rangle$ is an M-base, and $B_1 \in Z(p)$. Therefore $B_1 \in Z(q)$, and $q' \in B_1$, $q \in B_1$ which contradicts (b)II of Lemma 1.

Now define a map $h : P \to Z(M(P))$ setting $h(p) = Z(p)$ for each $p \in P$.

(vii) h is bijective. This follows immediately from the definition of h and by (vi).

(viii) The orthocomplemented posets $(P, \leq, 0, 1, {}')$ and $(Z(M(P)), \subseteq, \emptyset, M(P), {}^*)$ are orthoisomorphic. The fact that h is an orthoisomorphism is namely a consequence of (vi), (vii) and (1).

Remark. If for $p_1, p_2 \in P$ $p_1 \vee p_2$ resp. $p_1 \wedge p_2$ exists in P, then the following equalities hold:

$h(p_1 \vee p_2) = h(p_1) \vee h(p_2)$, $h(p_1 \wedge p_2) = h(p_1) \wedge h(p_2)$.

But it is necessary to warn. The operations $\vee$ and $\wedge$ in a poset $(Z(M(P)), \subseteq, \emptyset, M(P), {}^*)$ as long as they are defined may in general differ from the usual set-theoretical operations $\cup$ and $\cap$.

Proposition 3 [1]. Every zero-dimensional, completely regular topological T_1-space X of the total character $w(X) = \tau$ can be embedded homeomorphically in the Cantor cube $D^{\tau} = \prod_{s \in S} D_s$, where $D_s = \{0,1\}$, $s \in S$ are endowed as topological spaces with a discrete topology, and card $S = \tau$.

Proof: See [1].

Corollary. If $(P, \leq, 0, 1, ')$ is an orthocomplemented poset and if card $P = \tau$, then the space $(M(P), \mathcal{T})$ can be embedded homeomorphically in D^{τ}.

Proof: Clearly card $Z(M(P)) = \tau$. If $\mathcal{U}$ is a basis of clopen sets in $M(P)$ generated by $Z(M(P))$ as a subbasis of topology $\mathcal{T}$, then card $\mathcal{U} = \tau$. Therefore for the total character $W(M(P))$ of $M(P)$ we get $W(M(P)) \leq \tau$. Corollary follows now applying Theorem 1 and Proposition 3.

In a special case, when $(P, \leq, 0, 1, ')$ is a Boolean algebra, and $\mathcal{S}(P)$ the Stonean space of P, then the following assertion establishes the connection between the topological spaces $\mathcal{S}(P)$ and $M(P)$.

Proposition 4. Let $(P, \leq, 0, 1, ')$ be a Boolean algebra. Then the Stonean space $\mathcal{S}(P)$ of P is a compact subspace of the topological space $M(P)$.

Proof: Follows as a simple consequence of the fact that the topology of the Stone space $\mathcal{S}(P)$ is induced by the topology of $M(P)$.

Theorem 2. If the orthocomplemented posets $(P_1, \leq, 0_1, 1_1, ')$ $(P_2, \leq, 0_2, 1_2, *)$ are orthoisomorphic, then the corresponding topological spaces $(M(P_1), \mathcal{T}_1)$, $(M(P_2, \mathcal{T}_2)$ are homeomorphic.

Proof: Let $h: P_1 \longrightarrow P_2$ be an orthoisomorphism from P_1 on P_2. It is easy to show that B is an M-base in P_1 iff $h(B)$ is an M-base in P_2. Therefore the mapping h induces a mapping $\hat{h}: M(P_1) \rightarrow$ $\longrightarrow M(P_2)$. The bijectivity of h implies bijectivity of $\hat{h}$. Now if we denote by $Z_i(p)$, $p \in P_i$ the elements of subbases $Z_i(M(P_i))$ of topological spaces $(M(P_i), \mathcal{T}_i)$, $i = 1, 2$, then the following equality turns out to be valid:

(2) $\hat{h}^{-1}\ (Z_2(p)) = Z_1(h^{-1}(p)) \quad p \in P_2$

Now, if F is an element of the basis for closed subsets of the topological space $M(P_2)$, then there exists such $p_j \in P_2$, $j = 1, 2, \ldots, n$, that $F = \bigcup_{j=1}^{m} Z_2(p_j)$. According to (2) we obtain $\hat{h}^{-1}(F) = \hat{h}^{-1}(\bigcup_{j=1}^{m} Z_2(p_j)) = \bigcup_{j=1}^{m} \hat{h}^{-1}(Z_2(p_j)) = \bigcup_{j=1}^{m} Z_1(h^{-1}(p_j))$.
This implies that $\hat{h}^{-1}(F)$ is an element of basis for closed subsets in $M(P_1)$, and hence the continuity of $\hat{h}$. The continuity of $\hat{h}^{-1}$ can be shown analogically. The converse of the theorem may fail.

Example. Let be $X = \{1,2,3,4\}$, $P_1 = \{Y \in \exp X \mid \text{card } Y \notin \{1,3\}\}$, $Z = \{1,2,3,4,5,6,7,8\}$ and $P_2 = \{\emptyset, Z, \{1,2\}, \{3,4\}, \{5,6\}, Z \setminus \{1,2\}, Z \setminus \{3,4\}, Z \setminus \{5,6\}\}$.
Define the partial ordering and the orthocomplement on P_i, $i = 1,2$ as the inclusion relation and the set-theoretical comple-

ment respectively. It can be easily shown that $(P_1, \subseteq, \emptyset, X, ')$ and $(P_2, \subseteq, \emptyset, Z, ')$ are orthocomplemented posets. For the spaces $M(P_1)$, $M(P_2)$ it may be found that card $M(P_1)$ = card $M(P_2)$ = 4. So we can see that the spaces $M(P_1)$, $M(P_2)$ are discrete and homeomorphic. But the posets P_1, P_2 cannot be orthoisomorphic, because while P_2 contains three different mutually orthogonal elements, P_1 contains always only at most two mutually orthogonal elements.

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