COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
23,3 (1982)

SELF-DUAL SUBNORMAL OPERATORS

G. J. MURPHY

Abstract: A characterization of self-dual subnormal operators is given, and this characterization is shown to give quick proofs that certain classes of operators consist of self-dual subnormal operators.

Key words: Self-dual subnormal operator

Classification: 47B20

Recall that a subnormal operator is the restriction to an invariant subspace of a normal operator (all operators are understood to be bounded linear operators defined on Hilbert spaces). Every subnormal operator has a minimal normal extension N, and N is unique up to unitary equivalence [2]. Suppose then S is a subnormal operator on a Hilbert space H and N is a normal operator on a Hilbert space $K \supseteq H$ such that N is the minimal normal extension of S. Then relative to the decomposition $K = H \oplus H^{\perp}$ of K, N has operator matrix

$$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix}.$$

Now if S is a pure subnormal operator (i.e. S has no nonzero reducing subspace on which it is normal) then T is unique up to unitary equivalence and is called the dual of S (see, for example, [1]). S is said to be self-dual if S is unitarily equivalent to its dual T.

It is convenient to make the following definition - an operator S is <u>pure</u> if S has no non-zero reducing subspace on which S is normal.

We now give a simple characterization of self-dual subnormal operators which eliminates reference to the minimal normal extension.

$[X,Y]$ denotes the commutator $XY - YX$ for operators X and Y.

<u>Theorem 1</u>. Let S be a pure operator on a Hilbert space H. Then S is a self-dual subnormal operator if and only if there exists a normal operator A on H such that

$$[S^*,S] = AA^* \quad \text{and} \quad AS = S^*A.$$

<u>Proof</u>: Suppose first that S is a self-dual subnormal operator and

$$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix}$$

is its minimal normal extension on $H \oplus H$. Then for some unitary operator U on H, $T = USU^*$. But the equation $NN^* = N^*N$ implies

$$\begin{pmatrix} SS^* + XX^* & XT \\ T^*X^* & T^*T \end{pmatrix} = \begin{pmatrix} S^*S & S^*X \\ X^*S & X^*X + TT^* \end{pmatrix}.$$

Hence $[S^*,S] = XX^*$, $XT = S^*X$ and $[T^*,T] = X^*X$.

We define $A = XU$. Then $X = AU^*$, and $AS = XUS(U^*U) = (XT)U = (S^*X)U = S^*A$, i.e. $AS = S^*A$. Also $[S^*,S] = XX^* = AU^*(AU^*)^* = AA^*$. Finally A is normal, because

$$
\begin{aligned}
A^*A &= (XU)^*XU \\
&= U^*X^*XU \\
&= U^*[T^*,T]U \\
&= U^*((USU^*)^*USU^* - USU^*(USU^*)^*)U \\
&= U^*(US^*SU^* - USS^*U^*)U \\
&= [S^*,S] \\
&= AA^*
\end{aligned}
$$

Now to prove the converse, suppose we are given a normal operator A such that $[S^*,S] = AA^*$ and $AS = S^*A$, and we'll show this implies S is a self-dual subnormal operator.

Put

$$
N = \begin{pmatrix} S & A \\ 0 & S^* \end{pmatrix}
$$

Thus N is an operator on $H \oplus H$, and some trivial matrix calculations show

$$
N^*N = \begin{pmatrix} S^*S & S^*A \\ A^*S & A^*A+SS^* \end{pmatrix}
$$

$$
NN^* = \begin{pmatrix} SS^*+AA^* & AS \\ S^*A^* & S^*S \end{pmatrix}
$$

So from the relations $[S^*,S] = AA^*$ and $AS = S^*A$ we deduce that $NN^* = N^*N$, i.e. N is normal. Thus the proof will be concluded if we show N is the minimal normal extension of S.

Supposing it is not, we derive a contradiction:

(For notational convenience let K denote the space on which N acts and regard H as a subspace of K, so that $K = H \oplus H^{\perp}$.)

Now as N is not the minimal normal extension there exists a proper subspace M of K which reduces N, and M contains H

but is not equal to H. Thus N_M, the restriction of N to M, is normal.

Now $K = H \oplus H^{\perp} = (H \oplus M \ominus H) \oplus M^{\perp} = M \oplus M^{\perp}$.

Thus relative to the decomposition $K = H \oplus (M \ominus H) \oplus M^{\perp}$, N has operator matrix

$$N = \begin{pmatrix} S & X_1 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & N_2 \end{pmatrix}$$

and relative to the decomposition $K = M \oplus M^{\perp}$, N has operator matrix

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

Also since M is reducing for N, we must have N_1, N_2 normal. But we can also identify the operator matrix of N relative to the decomposition $K = H \oplus (M \ominus H) \oplus M^{\perp}$ as

$$N = \begin{pmatrix} S & X_1 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{(S^*)}} \\ 0 & & \end{pmatrix}$$

Hence identifying corresponding submatrices of the above 3 x 3 operator matrices we deduce that

$$S^* = \begin{pmatrix} X_2 & 0 \\ 0 & N_2 \end{pmatrix}$$

relative to the decomposition $(M \ominus H) \oplus M^{\perp}$.

Thus $S^* = X_2 \oplus N_2$ on the space $(M \ominus H) \oplus M^{\perp} = H^{\perp}$, and hence $S = X_2^* \oplus N_2^*$. This implies S is normal on the reducing subspace $M^{\perp}$ (since N_2 is normal) and hence $M^{\perp} = 0$ by the purity of S. Thus $M = K$, a contradiction. □

__Corollary 1.__ If S is a pure hyponormal operator and $[S^*,S]^{1/2} S = S^*[S^*,S]^{1/2}$ then S is a self-dual subnormal operator.

__Proof__: Take $A = [S^*,S]^{1/2}$.

__Corollary 2.__ If S is a pure isometry, S is a self-dual subnormal operator.

__Proof__: $S^* S = 1$ implies $[S^*,S] = 1 - SS^*$ is a projection, whence $[S^*,S]^{1/2} = 1 - SS^*$. Thus $[S^*,S]^{1/2}S = (1 - SS^*)S = 0 = S^*(1 - SS^*) = S^*[S^*,S]^{1/2}$. The result now follows by applying Corollary 1. □

__Corollary 3.__ A pure quasinormal operator S is a self-dual subnormal operator.

__Proof__: S has a commuting polar decomposition $S = U|S| = |S|U$, and as S is pure U is an isometry. Now $U^*|S| = |S|U^*$ also, so $S^*S - SS^* = U^*|S|U|S| - U|S|U^*|S| = |S|^2(U^*U - UU^*) = |S|^2(1 - UU^*)$. Hence $[S^*,S]^{1/2} = |S|(1 - UU^*)$.

We conclude $[S^*,S]^{1/2} S = |S|(S - UU^*S) = |S|(S - U|S|) = |S|(S - S) = 0$, and so also $S^*[S^*,S]^{1/2} = 0$. □

__Remarks.__ One could generalize Corollary 2 by stating that if S is a pure operator, $[S^*,S]$ is a projection, and $[S^*,S]S = S^*[S^*,S]$, then S is a self-dual subnormal operator.

The results in Corollaries 2 & 3 are not new, see [1] for example.

The condition given in Corollary 1 is not a necessary condition on an arbitrary pure operator that S be a self-dual subnormal. In [1] it is shown that the unilateral weighted shift

S with weights $(1/4, 1,1,1,\ldots)$ is a self-dual subnormal operator. But S does not satisfy the condition $[S^*,S]^{1/2} S = S^*[S^*,S]^{1/2}$. This is because $S^*S - SS^*$ is the diagonal operator with diagonal sequence $(1/4,3/4,0,0,\ldots)$, and hence $[S^*,S]^{1/2}$ is diagonal with sequence $(1/2, \frac{\sqrt{3}}{2},0,0,\ldots)$. Thus $[S^*,S]^{1/2}S e_0 = [S^*,S]^{1/2} e_1/4 = \frac{\sqrt{3}}{2}\frac{1}{4} e_1 \neq 0$ and $S^*[S^*,S]^{1/2}e_0 = 0$ (here as usual $e_0,e_1,e_2,\ldots$ denote the orthonormal basis for the Hilbert space). Hence $[S^*,S]^{1/2} S \neq S^*[S^*,S]^{1/2}$.

We conclude with a new characterization of the pure hyponormal operators which are self-dual subnormal operators.

<u>Theorem 2</u>. Let S be a pure hyponormal operator on the Hilbert space H. Then S is a self-dual subnormal operator if and only if there is a unitary operator U on H such that

$$U[S^*,S]^{1/2} S = S^*[S^*,S]^{1/2} U$$
$$\text{and} \quad U[S^*,S]^{1/2} = [S^*,S]^{1/2} U.$$

<u>Proof</u>: Suppose firstly that S is a self-dual subnormal. Then by Theorem 1 there is a normal operator A on H such that $AS = S^*A$ and $[S^*,S] = AA^*$. Now we can polar decompose $A = U|A| = |A|U$ where U is a unitary.

Hence $AA^* = |A|^2 = [S^*,S]$ implies $|A| = [S^*,S]^{1/2}$. Also $AS = S^*A$ implies $U[S^*,S]^{1/2} S = S^*[S^*,S]^{1/2} U$.

Conversely if we suppose that a unitary operator U exists for which $U[S^*,S]^{1/2} S = S^*[S^*,S]^{1/2} U$ and $U[S^*,S]^{1/2} = [S^*,S]^{1/2} U$, we simply put $A = U[S^*,S]^{1/2}$ and find that $[S^*,S] = AA^*$, $AS = S^*A$, and A is normal. Thus by Theorem 1, S is a self-dual subnormal operator. □

R e f e r e n c e s

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Dalhousie University, Halifax, N.S., Canada B3H 4H8

(Oblatum 17.2. 1982)