Werk

Label: Article

Jahr: 1982

PURL: https://resolver.sub.uni-goettingen.de/purl?316342866_0023|log39

PERIODIC SOLUTIONS FOR NONLINEAR PROBLEMS WITH STRONG RESONANCE AT INFINITY *)
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Abstract: In this paper we are looking for periodic solutions of the equations $-\ddot{x} = \nabla U(x,t)$. We suppose that the problem is asymptotically linear and that 0 belongs to the spectrum of linearized operator at infinity. We obtain multiplicity results. The proof of the theorem is based on a recent abstract theorem, that has been proved for a functional that satisfies a weaker condition than Palais-Smale condition.

Key words: Variational problem, Resonance, Periodic solutions

Classification: Primary 34C25
Secondary 47H15, 49G99

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0. __Introduction.__ The aim of this paper is to look for solutions $x(t) \in C^2(R,R^n)$ of the equations

$$(0.1) \quad \begin{cases} -\ddot{x} = \nabla U(x,t) \\ x(o) = x(T) \\ \dot{x}(o) = x(T) \end{cases}$$

where $T > 0$ is a given period, $U(x,t) \in C^2(R^n \times R, R)$, $U(x,t) = U(x,t+T)$ $\forall x \in R^n$ $\forall t \in R$.

The problem (0.1) has been studied by many authors under different assumptions on the function U. We refer to Benci [2] and Thews [5] for a rather complete bibliography. If the pro-

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blem is subquadratic (cf. [2]) multiplicity results for problem
(0.1) have been obtained in the non resonant case (i.e. if 0
does not belong to the spectrum of linearized operator at infi-
nity). It is well known that the solutions of (0.1) are the
critical points of the functional of the action in a suitable
function space. In the non resonant case this functional satis-
fies the well known Palais-Smale condition. The interest of the
resonant case lies in the fact that the Palais-Smale condition
is not always satisfied. Recently some techniques have been de-
veloped for studying non linear problems, having a variational
structure, with a "strong resonance" at infinity (cf. [1]).
Our purpose is to use these techniques for solving the problem
(0.1).

We denote by $U_{xx}(x,t)$ the Hessian matrix of $U(x,t)$ with
respect to the space variables and we assume that there exists

$$\lim U_{xx}(x,t) = M(t) \quad \text{as} \quad |x| \longrightarrow \infty \ \forall t \in [o,T]$$

where $M(t)$ is an $[n \times n]$ symmetric matrix with elements continu-
ous in $[o,T]$.

If we set

$$\nabla U(x,t) = M(t)x - \nabla V(x,t),$$

the problem (0.1) becomes

$$(0.2) \quad \begin{cases} -\ddot{x} = M(t)x - \nabla V(x,t) \\ x(o) = x(T) \\ \dot{x}(o) = \dot{x}(T). \end{cases}$$

We denote by $\mathcal{L}$ the self-adjoint realization in $L^2((o,T),R^n)$
of the operator $x \longrightarrow -\ddot{x} - M(t)x$ with periodic conditions. We as-
sume that

(I_1) $\nabla V(o,t) = o$ $\forall t \in R, \quad o \in \mathfrak{S}(\mathcal{L})$

(I_2) $V(x,t) \longrightarrow o$ as $|x| \longrightarrow \infty$ uniformly in $t \in R$

         $(\nabla V(x,t),x) \longrightarrow o$ as $|x| \longrightarrow \infty$ uniformly in $t \in R$.

We observe that if $(I_1),(I_2)$ hold, the problem (0.2) has a "strong resonance" at infinity.

We denote by $\mu(t)$ the smallest eigenvalue of $V_{xx}(o,t)$ and we also assume that

(I_3) $\mu = \inf_{[o,T]} \mu(t) > o$,

(I_4) there exists $\lambda_h \in \mathfrak{S}(\mathcal{L})$ $\lambda_h < o$ s.t. $\lambda_h + \mu > o$,

(I_5) $V(x,t) = V(-x,t)$ $\forall x \in R^n, \quad \forall t \in R$.

We consider the operator $x \longrightarrow -\ddot{x} - \nabla U(x,t)$ linearized at infinity and at origin and we set

$$L_\infty x = -\ddot{x} - M(t)x$$
$$L_o x = -\ddot{x} - M(t)x + V_{xx}(o,t)x.$$

We denote by m_∞ (resp. m_o) the maximal dimension of subspaces where L_∞ (resp. L_o) is negative semidefinite.

The following theorem holds:

<u>Theorem 0.1</u>. - If $(I_1),(I_2),(I_3),(I_4),(I_5)$ hold, then the problem (0.2) possesses at least m distinct pairs of nontrivial solutions with

$$m = m_\infty - m_o.$$

The proof of Theorem (0.1) is based on the abstract theorem (2.4) in [1].

1. <u>Notations and preliminaries</u>. We set $L^2 = L^2((o,T),R^n)$, $H^1 = H^1((o,T),R^n)$ and denote by

$$(\cdot,\cdot),\ (\cdot,\cdot)_{L^2},\ (\cdot,\cdot)_{H^1}$$

respectively the scalar product on R^n, L^2, H^1.

We set $H = \{u \in H^1 | u(o) = u(T)\}$ equipped with the scalar product

$$(u,v)_H = (u,v)_{H^1}.$$

If X is a real Banach space, we denote by X' its dual and by $\langle \cdot, \cdot \rangle$ the pairing between X' and X. In the sequel we shall use the unique symbol $\| \cdot \|$ for the norms in X and X'. If $R > o$ we set $B_R = \{u \in X | \ \|u\| \leq R\}$ and $S_R = \{u \in X | \ \|u\| = R\}$.

If $f \in C^1(X,R)$, we denote by $f'(u)$ the Fréchet derivative of f at $u \in X$.

We recall the following definition [1],[3], which is a weaker version of the well-known Palais-Smale condition.

<u>Definition 1.1</u>. - We shall say that $f \in C^1(X,R)$ satisfies the condition (I) in $]c_1,c_2[$, $(-\infty \leq c_1 < c_2 \leq +\infty)$, if

$$(I) \begin{cases} \text{(i)} & \text{every bounded sequence } \{u_k\} \subset f^{-1}(]c_1,c_2[), \text{ for} \\ & \text{which } \{f(u_k)\} \text{ is bounded and } f'(u_k) \rightarrow o, \text{ pos-} \\ & \text{sesses a convergent subsequence} \\ \text{(ii)} & \forall \ c \in]c_1,c_2[\ \exists \sigma ,R, \alpha > o \ \text{ s.t. } [c-\sigma ,c+\sigma] \subset \\ & \subset]c_1,c_2[\text{ and } \forall u \in f^{-1}([c-\sigma ,c+\sigma]), \|u\| \geq R: \\ & : \| f'(u)\| \ \|u\| \geq \alpha . \end{cases}$$

We shall need the following abstract theorem for a real functional f on a real Hilbert space M ([1], th. 2.4).

<u>Theorem 1.1</u>. - Suppose that $f \in C^1(M,R)$ satisfies the following properties:

f_1) f satisfies condition (I) in $]o,+\infty[$;

f_2) there exist two closed subspaces M^+, M^- of M, with codim $M^+<$
$< +\infty$, and two constant $c_\infty > c_0 > f(o)$ such that

    a) $f(u) > c_0$ $\forall u \in S_\rho \cap M^+$

    b) $f(u) < c_\infty$ $\forall u \in M^-$

f_3) f is even.

Then, if dim $M^- \geq$ codim M^+, f possesses at least m = dim M^--
-codim M^+ distinct pairs of critical points whose corresponding
critical values belong to $[c_0, c_\infty]$.

    2. - <u>Proof of the Theorem</u>. Standard arguments in the cal-
culus of variations show that the classical solutions of (0.2)
correspond to the critical points of the functional

(2.1) $f(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T (M(t)u(t),u(t))dt +$

        $+ \int_0^T V(u(t),t)dt$

defined on H. Clearly $f \in C^2(H,R)$ and $\forall u \in H$

(2.2) $\langle f'(u),h \rangle = \int_0^T (\dot{u},\dot{h})dt - \int_0^T (M(t)u,h)dt +$

        $+ \int_0^T (\nabla V(u,t),h)dt$ $\forall h \in H$

(2.3) $f''(u)[h,s] = \int_0^T (\dot{h},\dot{s})dt - \int_0^T (M(t)h,s)dt +$

        $+ \int_0^T (V_{xx}(u,t)h,s)dt$ $\forall h,s \in H.$

    We denote by $\beta(t)$ the largest eigenvalue of M(t) and by
I_n the identity matrix in R^n, and we set

          $\beta = \sup_{[o,T]} \beta(t)$ $M_1(t) = M(t) + I_n.$

    Let $a(u,v):H \times H \rightarrow R$ be the bilinear form defined by

$$a(u,v) = \int_o^T [(\dot{u},\dot{v}) + (u,v)] dt - \int_o^T (M_1(t)u,v) dt +$$
$$+ \beta \int_o^T (u,v) dt.$$

It is easy to verify that a is continuous and coercive (i.e. $a(u,u) \geq$ const $\|u\|_H^2$) on H. Then by the Lax-Milgram theorem there exists a unique bounded linear operator $S:H \to H$ with a bounded linear inverse S^{-1} such that

$$(Su,v)_H = a(u,v) \qquad \forall u,v \in H.$$

We set

$$\mathcal{D}(\mathcal{S}) = \{u \in H \mid Su \in L^2\}$$

and

$$\mathcal{S} = S_{|\mathcal{D}(\mathcal{S})}$$

$\mathcal{S}$ is a linear continuous self-adjoint operator with compact resolvent. Then $\sigma(\mathcal{S})$ consists of a positively divergent sequence of isolated eigenvalues with finite multiplicities. We denote by $s_0 < s_1 \ldots \ldots < s_j < \ldots$ the eigenvalues of $\mathcal{S}$ and by $\lambda_0 < \lambda_1 \ldots \ldots < \lambda_j < \ldots \ldots$ the eigenvalues of $\mathcal{L}$.

Obviously $\mathcal{L} = \mathcal{S} - \beta$ i, where $i:L^2 \to L^2$ is the identity map, $\lambda_j = s_j - \beta \quad \forall j$ and by (I_1) it follows that there exists k such that $\beta = s_k \in \sigma(\mathcal{S})$.

We denote by M_j the sequence of eigenspaces corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_j, \ldots$. If $m \geq 0$ is an integer number we set

$$H^-(m) = \bigoplus_{j \leq m} M_j$$

$$H^+(m) = \text{closure in H of the linear space spanned}$$
$$\text{by } \{M_j\}_{j \geq m}$$

Clearly $H^+(m) \cap H^-(m) = M_m$ and $H = H^-(m) \oplus H^+(m+1)$. For

every $u \in H$, we set

$$u = u^+ + u^- + u_0,$$

where $u^+ \in H^+(k+1)$, $u^- \in H^-(k-1)$, $u_0 \in M_k$.

Lemma 2.1. - There exist $\eta, \tau, \nu > 0$ such that

(i) $(Su, u^+)_{L^2} - \beta \| u^+ \|_{L^2}^2 \geq \eta \| u^+ \|_H^2 \qquad \forall u \in H$

(ii) $(Su, u^-)_{L^2} - \beta \| u^- \|_{L^2}^2 \geq -\tau \| u^- \|_H^2 \qquad \forall u \in H$

(iii) $\beta \| u^- \|_{L^2}^2 - (Su, u^-)_{L^2} \geq \nu \| u^- \|_H^2 \qquad \forall u \in H$

Proof. (i) $(Su, u^+)_{L^2} - \beta \| u^+ \|_{L^2}^2 = (Su^+, u^+)_{L^2} -$

$- \beta \| u^+ \|_{L^2}^2 = \sum_{j=k+1}^{\infty} (s_j - \beta) \| u_j \|_{L^2}^2 = \sum_{j=k+1}^{\infty} \frac{s_j - \beta}{s_j} s_j \| u_j \|_{L^2}^2 \geq$

$\geq \eta (Su^+, u^+)_{L^2} \geq \eta \| u^+ \|_H^2$

(ii) $(Su, u^-)_{L^2} - \beta \| u^- \|_{L^2}^2 = (Su^-, u^-)_{L^2} - \beta \| u^- \|_{L^2}^2 =$

$= \sum_{j=0}^{k-1} (s_j - \beta) \| u_j \|_{L^2}^2 \geq (s_0 - \beta) \sum_{j=0}^{k-1} \| u_j \|_{L^2}^2 = -\tau \| u^- \|_{L^2}^2 \geq$

$- \tau \| u^- \|_H^2$

(iii) $\beta \| u^- \|_{L^2}^2 - (Su, u^-)_{L^2} = \beta \| u^- \|_{L^2}^2 - (Su^-, u^-)_{L^2} =$

$= \sum_{j=0}^{k-1} (\beta - s_j) \| u_j \|_{L^2}^2 = \sum_{j=0}^{k-1} \frac{\beta - s_j}{s_j} s_j \| u_j \|_{L^2}^2 \geq \nu (Su^-, u^-)_{L^2} \geq$

$\geq \nu \| u^- \|_H^2.$

Lemma 2.2. - If $(I_1), (I_2)$ hold, the functional $f(u)$ defined by (2.1) satisfies the condition (I).

Proof. The proof is substantially analogous to the proof of Theorem (3.1) in [1]. It is only necessary to use the Lemma 2.1 and an obvious generalization of Lemma 3.2 in [1].

Lemma 2.3. - Suppose that $(I_1),(I_3),(I_4)$ hold, then there exist $\varphi > 0$ and $\gamma > 0$ such that

$$f(u) \geq f(o) + \gamma \qquad \forall u \in H^+(h) \cap S_\varphi .$$

Proof. We have $\forall u \in H$

$$(2.4) \quad f(u) = f(o) + \langle f'(o),u \rangle + f''(o) [u,u] + o(\|u\|_H^2) .$$

By (2.2) and by (I_1) we have $\forall u \in H$

$$(2.5) \qquad \langle f'(o),u \rangle = o.$$

By (2.3), $(I_3),(I_4)$ we have $\forall u \in H^+(h)$

$$f''(o) [u,u] = (Su,u)_{L^2} - \beta \|u\|_{L^2}^2 + \int_0^T (V_{xx}(o,t)u,u)dt =$$

$$= (Su^+,u^+)_{L^2} - \beta \|u^+\|_{L^2}^2 + \int_0^T (V_{xx}(o,t)u,u)dt \geq$$

$$\geq \sum_{j=h}^\infty s_j \|u_j\|_{L^2}^2 - (\beta - \mu) \|u^+\|_{L^2}^2 =$$

$$= \sum_{j=h}^\infty (s_j - \beta + \mu) \|u_j\|_{L^2}^2 .$$

There exist $t > h$ and $\delta > 0$ such that

$$s_j - \beta + \mu > \delta s_j \qquad \forall j > t,$$

then

$$\sum_{j=h}^\infty (s_j - \beta + \mu) \|u_j\|_{L^2}^2 = \sum_{j=h}^t (\frac{s_j - \beta + \mu}{s_j}) s_j \|u_j\|_{L^2}^2 +$$

$$+ \sum_{j=t+1}^\infty (s_j - \beta + \mu) \|u_j\|_{L^2}^2 \geq const \sum_{j=h}^t s_j \|u_j\|_{L^2}^2 +$$

$$+ \sum_{j=t+1}^\infty \delta s_j \|u_j\|_L^2 \geq const \sum_{j=h}^\infty s_j \|u_j\|_{L^2}^2 .$$

Then we have

(2.6) $f''(o) [u,u] \geq \text{const } \| u^+ \|^2_{L^2}.$

Finally, by (2.4),(2.5),(2.6) we have

$$f(u) \geq f(o) + \gamma \quad \text{with } \gamma > o.$$

Lemma 2.4. - There exists $\sigma > o$ such that

$$f(u) < \sigma \quad \forall u \in H^-(k).$$

Proof. Let

$$\lambda = \sup_{[o,T]} V(u,t),$$

then $\forall u \in H^-(k)$

$$f(u) = (Su^-,u^-)_{L^2} - \beta \| u^- \|^2_{L^2} + \int_0^T V(u,t)dt) \leq$$

$$\leq \sum_{j=o}^K (s_j - \beta) \| u_j \|^2_{L^2} + \lambda T \leq \lambda T.$$

Finally we can prove the Theorem (0.1).

Proof of Theorem 0.1. By Lemma 2.2, Lemma 2.3, Lemma 2.4 and by (I_5) we have that the functional f, defined by (2.1), satisfies $(f_1),(f_2),(f_3)$ of the Theorem (1.1). Hence, the problem (0.2) possesses at least

$$m = \dim (M_h \oplus \ldots \oplus M_k)$$

distinct pairs of nontrivial solutions.

Obviously

$$m = m_\infty - m_0.$$

3. - <u>A particular case</u>. We denote by $\mathcal{M}$ the self-adjoint realization in L^2 of the operator $x \rightarrow -\ddot{x}$ with periodicity con-

ditions, and we consider the particular case

$$M(t) = \alpha_k I_n, \quad \alpha_k = (\frac{2k\pi}{T})^2, \quad k = 0,1,\ldots\ldots$$

$\alpha_k \in \sigma(\mathcal{M})$ and the problem (0.2) becomes

$$(3.1) \quad \begin{cases} -\ddot{x}- \alpha_k x + \nabla V(x,t) = o \\ x(o) = x(T) \\ \dot{x}(o) = x(T) \end{cases}$$

If we assume that

$(I_4)'$ there exist $\alpha_h \leq \alpha_k$ s.t. $\alpha_h - \alpha_k + \mu > o$

we have that, if $(I_1),(I_2),(I_3),(I_4)',(I_5)$ hold, then the problem (3.1) possesses at least

$$m = \dim H^-(k) - \operatorname{codim} H^*(h)$$

distinct pairs of nontrivial solutions.

If we assume $\alpha_k = o$, we obtain the case studied by Thews [5].

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