COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
23,2 (1982)

GENERIC PROPERTIES OF VON KÁRMÁN EQUATIONS

Pavol QUITTNER

Abstract: The operator equation $f(w)=p$ connected with general boundary value problem for von Kármán equations is studied. It is proved that the singular sets $B=\{w;\ f'(w)$ is not surjective$\}$ and $f(B)$ are nowhere dense and that for every $p\notin f(B)$ the number of elements of $f^{-1}(p)$ is finite and odd. Also a generic result for the global structure of the solution set of equation $f(\lambda,w)=p$ /where λ is a bifurcation parameter/ is shown.

Key words: Fredholm map of index p, coercive, analytic, proper, compact.

Classification: 35J65

- - - - - - - - - - - - - - - - - -

1. NOTATION AND PRELIMINARIES

We restrict ourselves to consider the domain with infinitely smooth boundary /see Definition 1/, but the main results are available under some assumptions also for an angular domain whose boundary is piecewise of C^3 /see [1]/.

We shall use the notation and assumptions from [4] so that we just recall them.

Denote the partial derivatives by w_x, w_y, the outward normal derivative by $w_n = w_x n_x + w_y n_y$, the tangential derivative by $w_\tau = -w_x n_y + w_y n_x$.

Denote further

$$\Delta^2 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy} ,$$
$$[u,v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy} .$$

The boundary operators M,T are defined by

$$Mw = \nu\Delta w + (1-\nu)(w_{xx}n_x^2 + 2w_{xy}n_x n_y + w_{yy}n_y^2)$$
$$Tw = -(\Delta w)_n + (1-\nu)(w_{xx}n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy}n_x n_y)_\tau$$

where the Poisson constant $\nu \in \langle 0, \frac{1}{2})$.

For $u,v,\varphi \in W^{2,2}(\Omega)$ we define

$$(u,v)_{W_0^{2,2}} = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})\, dxdy ,$$
$$\|u\|_0 = ((u,u)_{W_0^{2,2}})^{\frac{1}{2}} ,$$
$$(u,v)_V = (u,v)_{W_0^{2,2}} + \nu\int_\Omega [u,v]\, dxdy ,$$
$$B(v;u,\varphi) = \int_\Omega (v_{xy}u_x\varphi_y + v_{xy}u_y\varphi_x - v_{xx}u_y\varphi_y - v_{yy}u_x\varphi_x)\, dxdy .$$

If $\varphi \in W_0^{2,2}(\Omega)$ we obtain $B(v;u,\varphi) = B(v;\varphi,u) = B(\varphi;u,v)$.

Definition 1. Let $\Omega \subset E_2$ be a simply connected bounded domain. Let there exist a one-to-one mapping θ of $\langle 0,R)$ onto $\partial\Omega$ defined by $\theta : t \mapsto (\omega_1(t), \omega_2(t))$ with the properties

$$\omega_i \in C^\infty(\langle 0,R)), \quad i=1,2,$$

$$\omega_{i+}^{(k)}(0) = \lim_{t \to R-} \omega_i^{(k)}(t) , \quad i=1,2, \quad k=0,1,2,\dots,$$

$(-\omega_2'(t), \omega_1'(t))$, $t \in \langle 0,R)$ is the unit vector of the inner normal to $\partial\Omega$.

Then we say that Ω is of the class C^∞.

Definition 2. Let $\sigma > 0$. Let the mapping

$$(x,y): \langle 0,R) \times \langle 0,\sigma\rangle \longrightarrow E_2$$

be defined by

$$x: (t,s) \longmapsto \omega_1(t) - s\,\omega_2'(t)$$
$$y: (t,s) \longmapsto \omega_2(t) + s\,\omega_1'(t) \ .$$

Denote by Ω_σ the image of $\langle 0,R) \times (0,\sigma)$ in this mapping.

Throughout the paper let

$$\Omega \in C^\infty, \quad \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \ , \quad \Gamma_i = \Theta(\gamma_i) \ , \quad i=1,2,3$$

where Θ is the mapping from Definition 1 and γ_i, $i=1,2,3$ are pairwise disjoint measurable subsets of $\langle 0,R)$.

By [4] there exists $\sigma_0 > 0$ such that the mapping (x,y) from Definition 2 is a one-to-one mapping of $\langle 0,R) \times \langle 0,\sigma_0\rangle$ onto $\overline{\Omega}_{\sigma_0}$. We shall suppose that

$$s_{xx}(s_y)^2 + s_{yy}(s_x)^2 - 2s_{xy}s_xs_y = 0 \qquad \text{on } \Gamma_2 \ .$$

Let us denote by V the closure of the set

$$\mathcal{V} = \{ u \in C^\infty(\overline{\Omega}) \ ; \quad u = u_n = 0 \quad \text{on } \Gamma_1 \ , \quad u = 0 \quad \text{on } \Gamma_2 \}$$

in the norm of $W^{2,2}(\Omega)$.

The functions k, m, r, ϕ, P specifying the boundary problem are supposed to fulfil /with arbitrary real numbers $p > 1$, $q > 2$/:

$k_2 \in L_p(\Gamma_2)$; $\quad k_2 \geq 0$ on Γ_2 ,

$k_{31} \in L_p(\Gamma_3)$; $\quad k_{31} \geq 0$ on Γ_3 ,

$k_{32} \in L_1(\Gamma_3)$; $\quad k_{32} \geq 0$ on Γ_3 ,

$m_2 \in L_p(\Gamma_2)$, $\quad m_3 \in L_p(\Gamma_3)$, $\quad r_3 \in L_1(\Gamma_3)$, $\quad P \in L_p(\Omega)$,

$\phi_0 \in W^{3-\frac{1}{q},q}(\partial\Omega)$, $\quad \phi_1 \in W^{2-\frac{1}{q},q}(\partial\Omega)$,

$$\phi_1 = \phi_0 = 0 \quad \text{on } \Gamma_3 .$$

Then there exists a function $F \in C^2(\bar{\Omega})$ which satisfies the conditions

$$F = \phi_0 \ , \qquad F_n = \phi_1 \qquad \text{on } \partial\Omega$$

/see [6]/.

Let us introduce the following bilinear forms:

$$a(w,\varphi) = \int_{\Gamma_2} k_2 w_n \varphi_n \, dS + \int_{\Gamma_3} (k_{32} w\varphi + k_{31} w_n \varphi_n) dS \ ,$$

$$((w,\varphi)) = (w,\varphi)_V + a(w,\varphi) \ .$$

We shall suppose

(1.1) $\qquad w \in V \ , \quad ((w,w)) = 0 \qquad \Longrightarrow \qquad w = 0 \ .$

Then $\|w\| = ((w,w))^{\frac{1}{2}}$ is an equivalent norm to $\|\cdot\|_{W^{2,2}}$ in V /see [3]/.

__Definition 3__. The couple $(w,\phi) \in V \times W^{2,2}(\Omega)$ is said to be **a variational** solution of the problem if

(1.2) $\qquad ((w,\varphi)) = B(w;\phi,\varphi) + \int_{\Omega} P\varphi dxdy + \int_{\Gamma_3} (r_3\varphi + m_3\varphi_n)\, dS + \int_{\Gamma_2} m_2 \varphi_n dS$

holds for each $\varphi \in V$,

(1.3) $\qquad (\phi,\Upsilon)_{W_0^{2,2}} = - B(w;w,\Upsilon)$ holds for each $\Upsilon \in W_0^{2,2}(\Omega)$,

(1.4) $\qquad \phi = \phi_0 \ , \quad \phi_n = \phi_1$ on $\partial\Omega$ in the sense of traces.

The sufficiently smooth variational solution defined above is the classical solution of the system of equations

$$\left.\begin{aligned} \Delta^2 w &= [w,\phi] + P \\ \Delta^2 \phi &= -[w,w] \end{aligned}\right\} \quad \text{on } \Omega$$

satisfying the boundary conditions

$$w = w_n = 0 \qquad \text{on } \Gamma_1 ,$$

$$w = 0 \ , \quad Mw + k_2 w_n = m_2 \qquad \text{on } \Gamma_2 ,$$

$$Mw + k_{31}w_n = m_3\,, \quad Tw + (w_x\phi_{y\tau} - w_y\phi_{x\tau}) + k_{32}w = r_3 \quad \text{on } \Gamma_3\,,$$

$$\phi = \phi_o\,, \quad \phi_n = \phi_1 \quad \text{on } \partial\Omega\,.$$

2. REFORMULATION OF THE PROBLEM

Let $w \in W^{2,2}(\Omega)$. Using the Hölder inequality and the continuous imbedding $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$ we obtain that $B_w : \psi \longmapsto B(w;w,\psi)$ is a continuous linear functional on $W_o^{2,2}(\Omega)$ so that by the Riesz theorem

$$(\exists!\ R(w) \in W_o^{2,2}(\Omega))\ (\forall\ \psi \in W_o^{2,2}(\Omega)) \qquad (R(w),\psi)_{W_o^{2,2}} = B(w;w,\psi)\ .$$

Similarly

$$(\exists!\ \tilde{F} \in W_o^{2,2}(\Omega))(\forall\ \psi \in W_o^{2,2}(\Omega)) \qquad (\tilde{F},\psi)_{W_o^{2,2}} = (F,\psi)_{W_o^{2,2}}\ ,$$

$$(\exists!\ C(w) \in V)(\forall\ \varphi \in V) \qquad ((C(w),\varphi)) = B(w;R(w),\varphi)\ ,$$

$$(\exists!\ L(w) \in V)(\forall\ \varphi \in V) \qquad ((L(w),\varphi)) = B(w;F-\tilde{F},\varphi)\ ,$$

$$(\exists!\ p \in V)(\forall\ \varphi \in V) \quad ((p,\varphi)) = \int_{\Omega} P\varphi\,dx\,dy + \int_{\Gamma_3}(r_3\varphi + m_3\varphi_n)\,dS + \int_{\Gamma_2} m_2\varphi_n\,dS.$$

Now we can reformulate the conditions (1.3) and (1.4) as

(2.1) $\qquad \phi = -R(w) + F - \tilde{F}\ .$

Substituting from (2.1) into (1.2) we obtain the equation

(2.2) $\qquad f(w) = p$

where

$$f : V \longrightarrow V : w \longmapsto f(w) = w + C(w) - L(w)\ .$$

The equation (2.2) is obviously equivalent to our problem.

3. PROPERTIES OF OPERATOR f

Lemma 1. The operators $C, L: V \to V$ are compact.

Proof. Let $\{w^n\} \subset V$ be bounded. We shall prove that $\{C(w^n)\}$ and $\{L(w^n)\}$ are relatively compact in V.

We may assume $w^n \rightharpoonup w$ in V, $w^n_x \rightharpoonup w_1$ and $w^n_y \rightharpoonup w_2$ in $W^{1,2}(\Omega)$ /since $\{w^n_x\}$, $\{w^n_y\}$ are bounded in $W^{1,2}(\Omega)$/. Using the compact imbeddings $W^{2,2}(\Omega) \subset W^{1,2}(\Omega)$ and $W^{1,2}(\Omega) \subset L^2(\Omega)$ one can easily prove $w_1 = w_x$, $w_2 = w_y$. By the compact imbedding $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$ and by the compactness of the operator $T: W^{1,2}(\Omega) \to L^2(\partial\Omega): u \mapsto u/_{\partial\Omega}$ we have $w^n \to w$ in $W^{1,4}(\Omega)$, $w^n_x/_{\partial\Omega} \to w_x/_{\partial\Omega}$, $w^n_y/_{\partial\Omega} \to w_y/_{\partial\Omega}$ in $L^2(\partial\Omega)$.

Thus $\quad \|R(w^n) - R(w)\|_0 = \sup\limits_{\Upsilon \in W^{2,2}_0(\Omega),\, \|\Upsilon\|_0 \le 1} |(R(w^n) - R(w), \Upsilon)_{W^{2,2}_0}| =$

$= \sup |B(w^n; w^n, \Upsilon) - B(w; w, \Upsilon)| = \sup |B(\Upsilon; w^n, w^n) - B(\Upsilon; w, w)| \le$

$\le \sup \int_\Omega (2\,|\Upsilon_{xy}|\,|w^n_x w^n_y - w_x w_y| + |\Upsilon_{xx}|\,|(w^n_y)^2 - w^2_y| + |\Upsilon_{yy}|\,|(w^n_x)^2 - w^2_x|)\,dxdy \to 0,$

since e.g.
$$\int_\Omega |\Upsilon_{xy}|\,|w^n_x w^n_y - w_x w_y|\,dxdy \le$$

$$\le \int_\Omega |\Upsilon_{xy}|\,(|w^n_y|\,|w^n_x - w_x| + |w_x|\,|w^n_y - w_y|)\,dxdy \le$$

$$\le \|\Upsilon\|_0\,(\|w^n\|_{W^{1,4}}\|w^n - w\|_{W^{1,4}} + \|w\|_{W^{1,4}}\|w^n - w\|_{W^{1,4}}) .$$

Similarly $\quad \|C(w^n) - C(w)\| = \sup\limits_{\varphi \in V,\, \|\varphi\| \le 1} |((C(w^n) - C(w), \varphi))| =$

$= \sup |B(w^n; R(w^n), \varphi) - B(w; R(w), \varphi)| \to 0 .$

Finally, $\quad \|L(w^n) - L(w)\| = \sup\limits_{\varphi \in V,\, \|\varphi\| \le 1} |B(w^n - w; F - \tilde{F}, \varphi)| \le$

$\le \sup |B(w^n - w; \tilde{F}, \varphi)| + \sup |B(w^n - w; F, \varphi)| .$

Clearly, $\quad \sup |B(w^n - w; \tilde{F}, \varphi)| = \sup |B(\tilde{F}; \varphi, w^n - w)| \to 0 .$

'sing the integration by parts we get $\sup |B(w^n - w; F, \varphi)| \to 0.$

Lemma 2. There exists a constant K such that for each $w \in V$ the following estimate holds

$$((C(w),w)) - |((L(w),w))| \geq -\frac{1}{2}\|w\|^2 - K \quad .$$

Proof. There exists a function $\xi \in C^\infty(\bar{\Omega})$ with the properties:

$$\left.\begin{aligned} \xi &= 1 \\ \xi_x &= \xi_y = 0 \end{aligned}\right\} \text{ on } \partial\Omega \,,$$

$$|B(w;\xi F,w)| \leq \frac{1}{2}\|w\|^2 \qquad \text{for each } w \in V$$

/see [4], Lemma 5/.

Using the Riesz theorem we get

$$(\exists !\ \widetilde{\xi F} \in W_o^{2,2}(\Omega))(\forall \tau \in W_o^{2,2}(\Omega)) \qquad (\widetilde{\xi F},\tau)_{W_o^{2,2}} = (\xi F,\tau)_{W_o^{2,2}} \quad .$$

Since $F-\tilde{F} = \xi F - \widetilde{\xi F}$, we have

$$((C(w),w)) - |((L(w),w))| = B(w;R(w),w) - |B(w;\xi F-\widetilde{\xi F},w)| \geq$$

$$\geq B(w;w,R(w)) - |B(w;\xi F,w)| - |B(w;w,\widetilde{\xi F})| \geq$$

$$\geq \|R(w)\|_o^2 - \frac{1}{2}\|w\|^2 - \|R(w)\|_o \cdot \|\widetilde{\xi F}\|_o =$$

$$= -\frac{1}{2}\|w\|^2 + \|R(w)\|_o(\|R(w)\|_o - \|\widetilde{\xi F}\|_o) \geq -\frac{1}{2}\|w\|^2 - \|\widetilde{\xi F}\|_o^2 \quad .$$

Corollary. The operator f is coercive.

Definition 4. Let X,Y be Banach spaces, $A: X \to Y$ a continuous linear mapping, $f: X \to Y$ a /nonlinear/ C^1 map.

The mapping A is said to be a Fredholm mapping of index p if $\operatorname{Im} A$ is closed, $\dim \operatorname{Ker} A < \infty$, $\operatorname{codim} \operatorname{Im} A < \infty$ and $p = \dim \operatorname{Ker} A - \operatorname{codim} \operatorname{Im} A$.

The map f is said to be a Fredholm map of index p if $f'(x)$ is a linear Fredholm mapping of index p for each $x \in X$.

The map f is said to be proper if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact.

Lemma 3. The operator f is a Fredholm map of index zero.

Proof. Let $w \in V$. Since L, C are compact analytic operators, their derivatives $L'(w)$, $C'(w)$ have to be compact mappings. Thus $f'(w) = Id - L'(w) + C'(w)$ is the compact perturbation of the identity and hence it is a linear Fredholm mapping of index 0.

Lemma 4. The operator f is proper.

Proof. Let $K \subset Y$ be compact, let us choose a sequence $\{w^n\} \subseteq f^{-1}(K)$. Since f is coercive, $\{w^n\}$ is bounded. According to Lemma 1 we may assume $C(w^n) \to p^1$, $L(w^n) \to p^2$. Further $\{f(w^n)\} \subseteq K$ so that we may assume $f(w^n) \to p \in K$. Thus $w^n = f(w^n) - C(w^n) + L(w^n) \to p - p^1 + p^2$ and hence $f^{-1}(K)$ is relatively compact. Since f is continuous, $f^{-1}(K)$ is closed.

4. MODIFIED SMALE'S THEOREM

Let X, Y be real Banach spaces, $U \subseteq X$ open, $M \subseteq U$. Let $f: U \to Y$ be a C^1 map. We shall denote the restriction of f to M by f/M. Further denote

$B(f/M) = \{x \in M;\ f'(x) \text{ is not surjective}\}$,

$\mathcal{O}(f/M) = \{y \in Y;\ (\forall x \in M \cap f^{-1}(y))\ f'(x) \text{ is surjective}\} = Y - f(B(f/M))$,

$B(f) = B(f/U)$, $\mathcal{O}(f) = \mathcal{O}(f/U)$.

Then $\mathcal{O}(f/M_1) \supseteq \mathcal{O}(f/M_2)$ for $M_1 \subseteq M_2$ and $y \in \mathcal{O}(f/M)$ for each $y \notin f(M)$.

Theorem 1. Let X, Y be real Banach spaces, $U_1, U_2 \subseteq X$ open subsets, $\overline{U}_1 \subseteq U_2$. Let $f: U_2 \to Y$ be a C^k /resp. real analytic/ Fredholm map of index $p \geq 0$, $p < k$. Let $f^{-1}(K)$ be relatively compact /in X/ whenever $K \subset Y$ is compact.

Then the set $\mathcal{O} = \mathcal{O}(f/\overline{U}_1)$ is a dense open subset of Y and for every $y_0 \in \mathcal{O}$ the set $f^{-1}(y_0) \cap U_1$ is a C^k /resp. analytic/ manifold of dimension p. If p=0 the set $f^{-1}(y_0) \cap U_1$ is finite /for $y_0 \in \mathcal{O}$/.

Proof. We shall prove that the set $\mathcal{O}$ is dense and open in Y; all remaining assertions follow from the implicit function theorem.

First we show that f is a closed mapping.
Let $Z \subseteq U_2$ be closed /in X/, let $x_n \in Z$, $f(x_n) \to y$. Since $\{x_n\}$ is relatively compact, we may assume $x_n \to x \in Z$. Then $f(x)=y$, $y \in f(Z)$. Consequently $f(Z)$ is closed.

Since $B(f/\overline{U}_1)$ is closed and f is a closed mapping, the set $\mathcal{O}$ is open.

Let us choose $y \in Y$. Then $K = f^{-1}(y) \cap \overline{U}_1$ is compact. Let $x \in K$. By [2] /see the proof of Theorem C.1.3./ there exists a neighbourhood U_x of x such that the set $\mathcal{O}(f/U_x)$ is dense. Let us choose $W_x \subset U_x$ a closed neighbourhood of x. Then the set $\mathcal{O}(f/W_x)$ is open /since $B(f/W_x)$ is closed and f is a closed mapping/ and dense /since $\mathcal{O}(f/W_x) \supseteq \mathcal{O}(f/U_x)$ /. Further choose an open set V_x such that $x \in V_x \subset W_x$. Since $K \subseteq \bigcup_{x \in K} V_x$, there exists a finite set $\{x_1, \dots, x_n\} \subseteq K$ such that $K \subseteq \bigcup_{i=1}^{n} V_{x_i}$. Let us denote $G = \bigcup_{i=1}^{n} V_{x_i}$. Since $\mathcal{O}(f/W_{x_i})$, $i=1,\dots,n$ is dense and open and $\mathcal{O}(f/G) \supseteq \bigcap_{i=1}^{n} \mathcal{O}(f/W_{x_i})$, the set $\mathcal{O}(f/G)$ is dense in Y.
One can easily prove that there exists a neighbourhood $\widetilde{U}$ of y such that $\widetilde{U} \cap f(\overline{U}_1 - G) = \emptyset$. Then $\widetilde{U} \cap \mathcal{O}(f/G) \subseteq \mathcal{O}$ and hence the set $\mathcal{O}$ is dense.

Lemma 5. Let the assumptions of Theorem 1 be fulfilled. Let $U_1=U_2=X$, $p=0$. Then $\operatorname{card} f^{-1}(y)$ /i.e. the number of elements of the set $f^{-1}(y)$/ is constant on every connected component of $\mathcal{O}$.

Proof. It is sufficient to prove that $\operatorname{card} f^{-1}(y)$ is a continuous function on $\mathcal{O}$.
Choose $y_0 \in \mathcal{O}$; let $f^{-1}(y_0) = \{x_1, \dots, x_N\}$. By the implicit function theorem there exists an open neighbourhood O_i of x_i /$i=1,\dots,N$/ such that f/O_i is a diffeomorphism. Thus $\operatorname{card} f^{-1}(y)$ is a lower semicontinuous function and it remains to show that it is also upper semicontinuous.
Let us suppose $z_n \notin \bigcup_{i=1}^{N} O_i$, $f(z_n) \to y_0$. We may assume $z_n \to z$. But then $f(z) = y_0$, $z \notin \bigcup_{i=1}^{N} O_i$, which contradicts the construction of O_i.

5. THE STRUCTURE OF THE SOLUTION SET

Theorem 2. Let $f: V \to V$ be the mapping defined in Section 2. Then $\mathcal{O} = \mathcal{O}(f)$ is a dense open subset of V and $\operatorname{card} f^{-1}(p)$ is finite, odd and locally constant for $p \in \mathcal{O}$.

Proof. According to Lemmas 3,4,5 and Theorem 1 it remains to prove that $\operatorname{card} f^{-1}(p)$ is odd /for $p \in \mathcal{O}$/.
Let $p \in \mathcal{O}$. For $\mu \in \langle 0,1 \rangle$ we define operators

$$f_\mu : V \to V : w \mapsto w + \mu(C-L)(w) .$$

By Lemma 2 there exists a constant K such that for every $w \in V$ and every $\mu \in \langle 0,1 \rangle$ the following estimate holds

$$((f_\mu(w), w)) \geq \tfrac{1}{2}\|w\|^2 - K .$$

Consequently, there exists an open bounded set U in V such that $p \in U$, $f^{-1}(p) \subseteq U$ and $p \notin f_\mu(\partial U)$ for every μ.
By the homotopy invariance property of the Leray-Schauder degree we have

$$\deg(f,U,p) = \deg(f_1,U,p) = \deg(f_0,U,p) = 1 .$$

Since $\deg(f,U,p) = \sum_{j=1}^{N} i(w_j)$, where $\{w_1,\dots,w_N\} = f^{-1}(p)$ and $i(w_j) = \pm 1$ $/j=1,\dots,N/$, we get that $N = \operatorname{card} f^{-1}(p)$ is an odd number.

Now let us consider /instead of (1.4)/ the following boundary conditions

$$(5.1) \qquad \Phi = \lambda\Phi_0 , \qquad \Phi_n = \lambda\Phi_1$$

/λ being a real number/.

The operator $f = f^\lambda$ connected with the boundary conditions (5.1) can be written in the form $f^\lambda = Id + C^\lambda - L^\lambda$, where $C^\lambda = C$, $L^\lambda = \lambda L$ and C,L are operators connected with the boundary conditions (1.4).

Let us define the following operator

$$g:\ V \times E_1 \longrightarrow V:\ (w,\lambda) \longmapsto f^\lambda(w) = w + C(w) - \lambda L(w) .$$

Theorem 3.

(i) The set $\mathcal{O}_M = \mathcal{O}(g/V\times\langle -M,M\rangle)$ is dense and open for any $M \in E_1$. For every $p \in \mathcal{O}_M$ the set $g^{-1}(p) \cap (V\times\langle -M,M\rangle)$ is an analytic relatively compact manifold of dimension 1.

(ii) $\mathcal{O}(g)$ is a residual set. For each $p \in \mathcal{O}(g)$ the set $g^{-1}(p)$ is a 1-dimensional analytic manifold and there exists a discrete set $D = D(p) \subset E_1$ such that the equation $f^\lambda(w) = p$ has only a finite number of solutions for any $\lambda \notin D$.

Proof.

(i) g is obviously a Fredholm map of index 1. By Lemma 2 we have

$$((C^{\lambda}(w),w)) - |((L^{\lambda}(w),w))| \geq -\frac{1}{2}\|w\|^2 - K_{\lambda} .$$

Thus for $|\lambda| \leq M$ we obtain

$$((C(w),w)) - |\lambda||((L(w),w))| \geq ((C(w),w)) - M|((L(w),w))| =$$

$$= ((C^{M}(w),w)) - |((L^{M}(w),w))| \geq -\frac{1}{2}\|w\|^2 - K_M ,$$

hence $g/V\times\langle -M,M\rangle$ is coercive /i.e. $\lim\limits_{\substack{|x|\to\infty \\ x\in V\times\langle -M,M\rangle}} \frac{(g(x),x)}{|x|} = +\infty$,

where $(\cdot,\cdot)$ is a scalar product in $V\times E_1$ and $|x| = (x,x)^{\frac{1}{2}}$ /. Now one can easily prove /analogously as in Lemma 4/ that $g/V\times\langle -M,M\rangle$ is proper. Using Theorem 1 with $U_1 = V\times(-M,M)$, $U_2 = V\times(-M-\varepsilon,M+\varepsilon)$, $\varepsilon>0$ we get our assertion.

(ii) $\mathcal{O}(g) = \bigcap\limits_{n=1}^{\infty}\mathcal{O}_n$, hence $\mathcal{O}(g)$ is a residual set.

$g^{-1}(p) = \bigcup\limits_{n=1}^{\infty}((V\times(-n,n))\cap g^{-1}(p))$, hence $g^{-1}(p)$ is 1-dimensional analytic manifold.

Let us consider the projection $\Pi: g^{-1}(p) \longrightarrow E_1 : (w,\lambda) \longmapsto \lambda$. Π is an analytic map, Π is proper. Using [9] for the maps of the form $\Pi\circ\Lambda$ /where $\Lambda: E_1 \longrightarrow g^{-1}(p)$ is a local description of the manifold $g^{-1}(p)$/ we get that the set $D = E_1 - \mathcal{O}(\Pi)$ is discrete. Our assertion now follows from the implicit function theorem.

Remark 1. The problem $g(w,\lambda) = p$ is often studied in the bifurcation theory. Theorem 3 shows that for generic p there is no bifurcation /cf. [7]/.

Remark 2. Let us choose $p_0 \in V$ and define the operator

$$h: V\times E_1\times E_1 \longrightarrow V : (w,\lambda,\mu) \longmapsto g(w,\lambda) + \mu p_0 .$$

Analogously as in Theorem 3 we get that $\mathcal{O}(h)$ is a residual set, for each $p \in \mathcal{O}(h)$ the set $h^{-1}(p)$ is an analytic manifold of dimension 2 and $h^{-1}(p) \cap (V \times K)$ is compact if $K \subset E_1 \times E_1$ is compact. Let us define the projection

$$\Pi : h^{-1}(p) \longrightarrow E_1 : (w, \lambda, \mu) \longmapsto \mu .$$

Then the set $E_1 - \mathcal{O}(\Pi)$ is discrete and for each $\mu \in \mathcal{O}(\Pi)$ the set $g^{-1}(p+\mu p_0)$ is an analytic manifold of dimension 1.

Let $p \notin \mathcal{O}(h)$. If there exists $\tilde{\mu} \in E_1$ such that $p+\tilde{\mu} p_0 \in \mathcal{O}(h)$, then we can repeat our considerations and we get again that $g^{-1}(p+\mu p_0)$ is an analytic manifold for generic μ.

6. THE SINGULAR SET B

<u>Theorem 4</u>. The set $B = B(f)$ is nowhere dense.

Proof. Since $\mathcal{O}$ is nonempty and f is surjective, there exists $w_0 \notin B$. Choose $w \in V$ and define /for $\varkappa \in E_1$/

$$T(\varkappa) = L - C'(w_0 + \varkappa(w - w_0)) \quad .$$

Obviously

$$w_0 + \varkappa(w - w_0) \in B \quad \Longleftrightarrow \quad 1 \text{ is an eigenvalue of } T(\varkappa) .$$

T is an analytic mapping of E_1 into the set of compact linear mappings on V and 1 is not an eigenvalue of the operator $T(0)$. By [5] /Theorem VII.1.9/ the set

$$\{\varkappa \in E_1 ;\ 1 \text{ is an eigenvalue of } T(\varkappa)\}$$

is discrete. Thus B is nowhere dense.

<u>Corollary</u>. The set $f^{-1}(f(B))$ is nowhere dense.

Proof. Choose $w \in V$ and its open neighbourhood U. Since B is nowhere dense, there exists $v \in U-B$. By the implicit function theorem there exists an open neighbourhood $\tilde{U}$ of v $/\tilde{U} \subseteq U/$ such that $f/\tilde{U}$ is a diffeomorphism. Since $f(\tilde{U})$ is open, there exists $p \in f(\tilde{U}) \cap \mathcal{O}$. Let $z \in f^{-1}(p) \cap \tilde{U}$. Then $z \notin f^{-1}(f(B))$ and $z \in U$.

Remark 3. If the operator $(Id-L)$ is invertible then Theorem 4 can be proved in an elementary way:

We have $\quad f'(\lambda w) = Id - L + \lambda^2 C'(w)$,

consequently

$$\begin{aligned} \lambda w \in B \quad & \Longleftrightarrow \quad (\exists v \neq 0) \quad (Id-L)v + \lambda^2 C'(w)v = 0 \\ & \Longleftrightarrow \quad (\exists v \neq 0) \quad v + \lambda^2 (Id-L)^{-1} C'(w) v = 0 \\ & \Longleftrightarrow \quad -\frac{1}{\lambda^2} \text{ is an eigenvalue of } (Id-L)^{-1} C'(w) . \end{aligned}$$

Since $(Id-L)^{-1} C'(w)$ is compact, the set $\{\lambda \in E_1 ;\ \lambda w \in B\}$ is discrete.

REFERENCES

[1] Franců J.: On Signorini problem for von Kármán equations /The case of angular domain/. Aplikace matematiky 24 (1979), 355-371.

[2] Geba K.: "The Leray Schauder degree and framed bordism" in La théorie des points fixes et ses applications à l'analyse. Séminaire de Mathématiques Supérieures 1973, Presses de l'Université de Montreal 1975.

[3] Hlaváček I., Naumann J.: Inhomogeneous boundary value problems for the von Kármán equations, I. Aplikace matematiky 19 (1974), 253-269.

[4] John O., Nečas J.: On the solvability of von Kármán equations. Aplikace matematiky 20 (1975), 48-62.

[5] Kato T.: Perturbation theory for linear operators. Springer-Verlag, Berlin - Heidelberg - New York, 1980.

[6] Nečas J.: Les méthodes directes en théorie des équations elliptiques. Academia, Prague, 1967.

[7] Saut J.C., Temam R.: Generic properties of Navier-Stokes equations: genericity with respect to the boundary values. Indiana Univ. Math. J. 29 (1980), 427-446.

[8] Smale S.: An infinite dimensional version of Sard's theorem. Amer. J. Math. 87 (1965) , 861-866.

[9] Souček J., Souček V.: The Morse Sard theorem for real analytic functions. Comment. Math. Univ. Carolinae 13 (1972), 45-51.

Author's address: Pavol Quittner, Matematicko-fyzikální fakulta KU, Sokolovská 83, 186 00 Praha 8.

(Oblatum 4.2. 1982)