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A NOTE ON CLOSED N-CELLS IN $\mathbb{R}^N$

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Abstract: In the paper the cohomological property of immersion of closed n-cells is given.

Key words: Closed n-cell, Čech cohomology theory, Alexander duality.

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§ 1. Introduction. In this paper some homological properties of closed n-cells will be discussed.

Let U and V be domains in $\mathbb{R}^N$, $U \subset V$, such that $\bar{U}$ and $\bar{V}$ are closed N-cells (i.e. sets homeomorphic to B^N, see [3]). One can show that in this situation $\bar{U}$ is a deformation retract of $\bar{V}$. It is an easy consequence of the fact that $\bar{U}$ and $\bar{V}$ are absolute retracts. But there is the second natural problem: Is the set ∂U a deformation retract of $\bar{V}-U$? In the polyhedral case there is a simplicial deformation. But in general case it seems to be a difficult problem.

The following statement is closely related with our question. We will prove it making use of the usual methods of algebraic topology.

Theorem. Let U and V be domains in $\mathbb{R}^N$ as above. Let $M \subset \bar{U}$ be a set which contains ∂U. Let us denote $R = \overline{V} - U$ and $J = R \cup M$. Then

$$\check{H}^*(J) = \check{H}^*(M)$$

$\check{H}^*(X)$ denotes here the Čech cohomology group of X with integral coefficients (for Čech cohomology see [1], [3]).

§ 2. Proof of the Theorem. Let us put

$$B^N = B = \{x = (x_1, \dots, x_n) \in \mathbb{R}^N | x_1^2 + \dots + x_N^2 < 1\}$$

$$S^{N-1} = S = \partial B$$

and assume $V = B$.

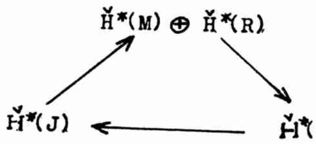
We can transform the general case to a special one using the homeomorphism of $\overline{B}$ with $\overline{V}$.

i) It is easy to see $\mathbb{R}^N - R \cong B \dot{\cup} (\mathbb{R}^N - \overline{B})$ ($\dot{\cup}$ - the topological sum). We get from the Alexander Duality Theorem (see [1])

$$\check{H}^{n-q-1}(R) \cong \tilde{H}_q(B \dot{\cup} (\mathbb{R}^N - \overline{B})) \cong \tilde{H}_q(\mathbb{R}^N - S) \cong \check{H}^{n-q-1}(S)$$

Hence R is a cohomological $(N-1)$-sphere.

ii) The couple (R,M) is excisive in the Čech cohomology theory (see [3] and [1]). Hence there is the Mayer-Vietoris exact triangle



(MV)

because $R \cap M = \partial U$ and $R \cup M = J$ by the definition. The Theorem can be obtained, if we use in (MV) the results of i).

Q.E.D.

Note that the Theorem can be proved for $N = 2$ without the explicit use of Alexander Duality Theorem.

Let R_i be the sequence (maybe finite) of components of $R - \partial U$. It is possible to show that $\overline{R}_i$ are Jordan domains and, by Schönflies Theorem, closed 2-cells. We can prove $J = M \cup_i \overline{R}_i$ and our Theorem follows by the continuity of the Čech cohomology theory.

References

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