COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
23,2 (1982)

A NOTE ON THE JOINT SPECTRUM IN COMMUTATIVE BANACH ALGEBRAS

Vladimír MULLER

Abstract: We characterize the part of the joint spectrum in a commutative Banach algebra which is always contained in the joint approximative spectrum.

Key words: Banach algebras, joint spectrum, joint approximative spectrum.

Classification: 46J20

Let A be a commutative Banach algebra with unit, $x_1,\dots,x_n \in A$ a finite family of elements of A. As usual, the joint spectrum of $x_1,\dots,x_n$ is defined by

$$\sigma(x_1,\dots,x_n) = \{[\hat{x}_1(M),\dots,\hat{x}_n(M)] \in \mathbb{C}^n,\ M \in \mathfrak{M}(A)\}$$

where $\mathfrak{M}(A)$ is the maximal ideal space of A and $\hat{x}$ is the Gelfand transform of $x \in A$. It is easy to see that $(\lambda_1,\dots,\lambda_n) \in \sigma(x_1,\dots,x_n)$ if and only if there exists a proper ideal in A containing $x_i - \lambda_i$ $(i=1,\dots,n)$. As in [1] we define the joint approximative spectrum of $x_1,\dots,x_n$ by $\tau(x_1,\dots,x_n) = \{(\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$, there exists a sequence $\{b_k\}_{k=1}^{\infty} \subset A$ such that $\lim_{k\to\infty} \sum_{i=1}^{n} |b_k(x_i - \lambda_i)| = 0\}$. Obviously $\tau(x_1,\dots,x_n) \subset \sigma(x_1,\dots,x_n)$.

For n=1, it is well known that the topological boundary of the spectrum is always contained in the approximative spectrum, $\partial\sigma(x_1) \subset \tau(x_1)$. For $n \geq 2$, this is no longer true. The simplest example is the algebra B of all functions holomorphic in the open bidisc $D_2 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2, |\lambda_1| < 1, |\lambda_2| < 1\}$ and continuous on the boundary. If we take $x_1, x_2 \in B$, $x_1(t_1,t_2) = t_1$, $x_2(t_1,t_2) = t_2$ then it is easy to see that $\sigma(x_1,x_2) = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2, |\lambda_1| \leq 1, |\lambda_2| \leq 1\}$, $\partial\sigma(x_1,x_2) = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2$, either $|\lambda_1| = 1$ or $|\lambda_2| = 1\}$ but $\tau(x_1,x_2) = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2, |\lambda_1| = 1$ and $|\lambda_2| = 1\}$.

In this paper we give an answer to a natural question which part of the joint spectrum is always contained in the joint approximative spectrum. This question was investigated already in [3]. The present result, however, differs from that of [3] in two points: 1) the proof is different, 2) in [3] it is explicitly stated only that the joint approximative spectrum is always non-empty (it is possible, however, to obtain in the same way the result which we present here).

The proof of Theorem 1 is based on the result of [2] (in an equivalent formulation): Let $x_1,\dots,x_n \in A$, $(\lambda_1,\dots,\lambda) \notin \tau_A(x_1,\dots,x_n)$. Then there exists a commutative superalgebra $B \supset A$ such that $(\lambda_1,\dots,\lambda_n) \notin \sigma_B(x_1,\dots,x_n)$ i.e. $\tau_A(x_1,\dots,x_n) = \bigcap_{B \supset A} \sigma_B(x_1,\dots,x_n)$.

Let K be a non-empty compact subset of $\mathbb{C}^n$. Denote $\tilde{A}_K$ the norm closure of the algebra of all functions holomorphic in some neighbourhood of the set K with the norm $|f| = \sup_{\mu \in K} |f(\mu)|$ (we identify two functions whenever they coincide on K). Then the Shilov boundary $\Gamma(\tilde{A}_K)$ of the function algebra $\tilde{A}_K$ may be identified with a subset of K, $\Gamma(\tilde{A}_K) \subset K \subset \mathcal{M}(\tilde{A}_K)$ (see e.g. [4]) and for any n-tuple $(\lambda_1,\dots,\lambda_n) \in \Gamma(\tilde{A}_K) \subset K$ and any neighbourhood U of $(\lambda_1,\dots,\lambda_n)$ in K there exists a function $f \in \tilde{A}_K$ satisfying $\sup_{\mu \in U} |f(\mu)| > \sup_{\mu \in K-U} |f(\mu)|$.

<u>Theorem 1</u>: Let B be a commutative Banach algebra with unit, $x_1,\dots,x_n \in B$, $\sigma_B(x_1,\dots,x_n) = K \subset \mathbb{C}^n$. Then $\tau(x_1,\dots,x_n) \supset \Gamma(\tilde{A}_K)$.

Proof. Suppose on the contrary $(\lambda_1,\dots,\lambda_n) \in \Gamma(\tilde{A}_K) \subset K$ and $(\lambda_1,\dots,\lambda_n) \notin \tau_B(x_1,\dots,x_n)$. By [2] there exists a commutative superalgebra $C \supset B$ such that $(\lambda_1,\dots,\lambda_n) \notin \sigma_C(x_1,\dots,x_n)$. As the joint spectrum is a compact set there exists a neighbourhood U of $(\lambda_1,\dots,\lambda_n)$ such that $U \cap \sigma_C(x_1,\dots,x_n) = \emptyset$. Since $(\lambda_1,\dots,\lambda_n) \in \Gamma(\tilde{A}_K)$ there exists a function $\tilde{f} \in \tilde{A}_K$ satisfying $\sup_{\mu \in U} |\tilde{f}(\mu)| > \sup_{\mu \in K-U} |\tilde{f}(\mu)|$. So we can find also a function f holomorphic in some neighbourhood of K such that $\sup_{\mu \in U} |f(\mu)| > \sup_{\mu \in K-U} |f(\mu)|$.

Consider the element $y = f(x_1,\dots,x_n) \in B \subset C$. By the spectral mapping theorem (see e.g. [4]) we have for the spectral radii of y in the Banach algebras B and C

$$r_B(y) = \sup_{(\mu_1,\dots,\mu_n) \in \sigma_B(x_1,\dots,x_n)} |f(\mu_1,\dots,\mu_n)| = \sup_{(\mu_1,\dots,\mu_n) \in K} |f(\mu_1,\dots,\mu_n)| >$$

$$> \sup_{(\mu_1,\dots,\mu_n) \in K-U} |f(\mu_1,\dots,\mu_n)| \geq \sup_{(\mu_1,\dots,\mu_n) \in \sigma_C(x_1,\dots,x_n)} |f(\mu_1,\dots,\mu_n)| =$$

$$= r_C(y).$$

So we have $r_B(y) > r_C(y)$, a contradiction with the fact that the spectral radius does not depend on the considered algebra, $r_B(y) = r_C(y) = \lim_{k \to \infty} |y^k|^{1/k}$.

Corollary: Let $x_1,\dots,x_n$ be elements of a subalgebra A of a commutative Banach algebra B. Then

$\hat{\sigma}_A(x_1,\dots,x_n) = \hat{\sigma}_B(x_1,\dots,x_n)$ where $\hat{\sigma}(x_1,\dots,x_n)$ denotes the polynomially convex hull of the joint spectrum.

Proof: We have $\tau_A(x_1,\dots,x_n) \subset \tau_B(x_1,\dots,x_n) \subset$ $\subset \sigma_B(x_1,\dots,x_n) \subset \sigma_A(x_1,\dots,x_n)$ and the polynomially convex hulls of $\tau_A(x_1,\dots,x_n)$ and $\sigma_A(x_1,\dots,x_n)$ coincide by Theorem 1.

Remark: For n=1, $\Gamma(\tilde{A}_K) = \partial K$. So the well-known inclusion $\partial\sigma(x_1) \subset \tau(x_1)$ follows from Theorem 1.

R e f e r e n c e s

[1] R. HARTE: Spectral mapping theorems, Proc. Roy. Irish Ac. ser. A vol. 72(1972), 89-107.

[2] V. MÜLLER: Non-removable ideals in commutative Banach algebras, Studia Math. 74 (in print).

[3] Z. SŁODKOWSKI, W. ŻELAZKO: On joint spectra of commuting families of operators, Studia Math. 50(1974),127-148.

[4] W. ŻELAZKO: Banach algebras, Elsevier/PWN 1973.

Matematický ústav ČSAV, Žitná 25, Praha 1, Czechoslovakia

(Oblatum 21.1. 1982)