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ON A CLASS OF DISTRIBUTIVE STEINER QUASIGROUPS

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Abstract: Let $n \geq 4q$ be a positive integer. Then there exist at least $\binom{q}{3}$ non-isomorphic distributive Steiner quasigroups of order 3^n.

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Recently, several papers devoted to the theory of distributive Steiner quasigroups appeared. The purpose of this short note is to find some estimates for the number of isomorphism classes of finite distributive Steiner quasigroups. The result is based on a construction introduced in [3].

A distributive Steiner quasigroup is a groupoid satisfying the identities $x.xy=y$, $xy=yx$ and $x.yz=xy.xz$. Let Q be a distributive Steiner quasigroup. Denote by $M(Q)$ the set of all ordered pairs (a,b) such that $a,b \in Q$ and the subgroupoid of Q generated by $\{a,b,c,d\}$ is medial (i.e. satisfies the identity $xy.uv=xu.yv$) for all $c,d \in Q$. It is well known that $M(Q)$ is a congruence of Q and we put $m(Q)=\mathrm{card}\ Q/M(Q)$. This cardinal number is said to be the mediality index of Q. It is clear that $m(Q\times P)=m(Q)m(P)$ whenever Q and P are distributive

Steiner quasigroups.

Let G be a (additively written) group of exponent 3 and let $S=G\times(1,2,3)$. We define a binary operation on S as follows: $(x,1)(y,2)=(y,2)(x,1)=(x+y,3)$, $(x,1)(y,3)=(y,3)(x,1)=(-x+y,2)$, $(x,2)(y,3)=(y,3)(x,2)=(y-x,1)$, $(x,i)(y,i)=(x-y+x,i)$ for all $x,y\in G$ and $i=1,2,3$. We obtain thus a groupoid which will be denoted by G(3).

Proposition 1. For any group G of exponent 3, G(3) is a distributive Steiner quasigroup.

Proof. By the definition, G(3) is commutative and it suffices to check the other two identities, namely $a.ab=b$ and $a.bc=ab.ac$. Further, for any $i=1,2,3$, the set $G\times\{i\}$ is a subgroupoid of G(3), isomorphic to the core of G (which is clearly a distributive Steiner quasigroup). Therefore, we can confine ourselves to the case in which a,b,c have not the same first coordinates. We have, for example, $(x,1)[(x,1)(y,2)]=(x,1)(x+y,3)=(y,2)$ and $(x,1)[(y,1)(z,2)]=(x,1)(y+z,3)=(-x+y+z,2)=(x-y+x,1)(x+z,3)=[(x,1)(y,1)][(x,1)(z,2)]$. The remaining cases are similar.

Proposition 2. Let G be a non-commutative group of exponent 3 with the center Z(G). Then the quasigroup G(3)/M(G(3)) is isomorphic to (G/Z(G))(3). Moreover, $m(G(3))=3[G:Z(G)]$.

Proof. First, it is easy to observe that $((x,1)(y,1))\in M(S)$, $S=G(3)$, iff $x-y\in Z(G)$. Since the translations of S are automorphisms and M(S) is invariant, this statement remains true for $i=2,3$, and hence, for each $i=1,2,3$, $((x,i),(y,i))\in M(S)$ iff $x-y\in Z(G)$. Now, we show that such pairs exhaust the set M(S). Suppose, on the contrary, that $((x,i),(y,j))\in M(S)$

for $i \neq j$. Using the same arguments as above, we can restrict ourselves to the case $i=1$, $j=2$. We have $[(a,1)(b,2)][(c,1)(d,1)] = [(a,1)(c,1)][(b,2)(d,1)]$ for all $c,d \in G$. Hence $d-c+a=a-c+d$ and, for $c=0$, $a \in Z(G)$. Thus $d-c=-c+d$ and G is abelian, a contradiction. Now, we define a mapping f of $S/M(S)$ into $(G/Z(G))(3)$ by $f:(a,i) \mid M(S) \longrightarrow (a+Z(G),i)$. It is easy to show that f is an isomorphism of the quasigroups. The rest is clear.

Proposition 3. Let G be a group of exponent 3. Then G(3) is nilpotent of the same class of G. In particular, G(3) is nilpotent of class at most 3.

Proof. If G is abelian, a direct calculation shows that G(3) is medial and the result is verified in this case. Suppose that G is non-commutative and denote by n its nilpotency class. Put $G_1=G$, $G_{i+1}=G_i/Z(G_i)$ and $M_i = M(G_i(3))$ for $i=1, 2,\dots,n+1$. Applying repeatedly Proposition 2, we see that $G_i(3)/M_i$ is isomorphic to $G_{i+1}(3)$ for $i=1,\dots,n$, while $G_{n+1}(3)/M_{n+1}$ is medial, since G_{n+1} is abelian. Thus the nilpotency class of G(3) is just n. The rest follows from [2, Lemma 5.3].

Proposition 4. There exist three non-isomorphic distributive non-medial Steiner quasigroups A,B,C of order 3^6 and mediality indices 3^5, 3^4, 3^3, resp.

Proof. Let G be the group of order 3^5 defined by the relations (1.17) of [9] for n=3. Clearly, Z(G) has order 3, and so $m(A)=3^5$ for $A=G(3)$. Similarly, using the group G defined in [9, pg. 114] (for p=3 and $\alpha_1(j)=0$), we obtain a distributive Steiner quasigroup B of order 3^5 and mediality index 3^4. Finally, we put $C=S\times T$ where S is a non-medial distributive Steiner quasigroup of order 81 and T a medial distributive Steiner quasigroup of order 9 (see [6]).

Proposition 5. Let $n \geq 4q$ be a positive integer. Then there exist at least $\binom{q}{3}$ pair-wise non-isomorphic distributive Steiner quasigroups of order 3^n.

Proof. Let A,B,C be the quasigroups defined in the preceding proposition. We have seen that C is the direct product $S\times T$, while A and B are directly irreducible. For any ordered triple (a,b,c) of natural numbers such that $a+b+c \leq q$, we denote by $S(a,b,c)$ the direct product of a copies of A, b copies of B, c copies of C and a medial Steiner quasigroup of order $3^{n-6(a+b+c)}$. Obviously, card $S(a,b,c)=3^n$. Now, suppose that $S(a,b,c)$ and $S(a',b',c')$ are isomorphic. It is known (see [5]) that a Steiner quasigroup has a unique decomposition into a direct product of irreducible quasigroups. Since both A and B are irreducible, $a=a'$ and $b=b'$. Moreover, S is irreducible and we have $c=c'$. We have proved that the quasigroups $S(a,b,c)$ are pair-wise non isomorphic and the result follows easily.

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