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THE EPIS OF POS (Z)
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Abstract: In [4] and [5] J. Meseguer conjectured that in $\text{POS}(\omega)$ epis are exactly the dense maps. In [3] D. Lehmann and A. Pasztor gave an example of an epi which is not dense. This paper provides the exact characterization of all epis of $\text{POS}(Z)$, for arbitrary Z .

Key words: Epimorphism, Z -complete poset.

Classification: 06A10, 18A20, 18B99, 68A05

1. Introduction. For an arbitrary subset system Z (see [1]) let $\text{POS}(Z)$ be the category of Z -complete posets (i.e. posets in which any Z -set has a sup) and Z -continuous maps (i.e. maps preserving the sups of Z -sets). If $X \subseteq P$ and $\underline{P} \in |\text{POS}(Z)|$ then $\text{cl}(X)$ is the least subset Y of P which contains X and in which every Z -set has its sup (which has to exist in P) in Y . A map $f: \underline{P} \rightarrow \underline{Q} \in \text{Mor } \text{POS}(Z)$ is dense if $\underline{Q} = \text{cl}(f(\underline{P}))$. In [4] and [5] J. Meseguer conjectured that in $\text{POS}(\omega)$ epis are exactly the dense maps (and hence extremal monos coincide with full monos). In [3] D. Lehmann and myself gave a counterexample to this conjecture by constructing an epi which is not even dense. What makes it to be an epi?

In order to answer this question let us consider the following domain \underline{D} of figure 1, described in Meseguer [4]. Then let $B := \{b_n : n \in \omega\}$. Meseguer proved that for any

$\varphi, \psi: \underline{D} \rightarrow \underline{P} \in \text{MorPCS}(\omega)$, if $\varphi \upharpoonright B = \psi \upharpoonright B$, then $\varphi(a_\omega) = \psi(a_\omega)$ (since e.g. $\varphi(a_\omega) \geq \varphi(b_n) = \psi(b_n) \geq \psi(a_n) \forall n \in \omega$ and hence by $\psi(a_\omega) = \sup(\psi(a_n))_{n \in \omega}$ we get $\varphi(a_\omega) \geq \psi(a_\omega)$). But notice that $a_\omega \notin \text{cl}(B)$.

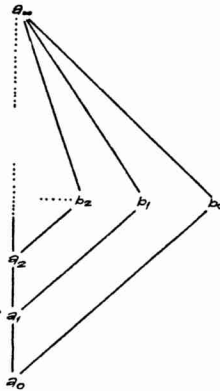


figure 1: domain \underline{D}

This leads to the counterexample, which is an embedding i of $B := \{(w, b) : w \in \omega^+\}$ into the domain \underline{E} of figure 2. Since every element of $E-B$ plays the role of the a_ω of figure 1, we get that for any $\varphi, \psi: \underline{E} \rightarrow \underline{P}$, $\varphi \upharpoonright B = \psi \upharpoonright B$ implies $\varphi = \psi$, which makes i to be an epi in $\text{PCS}(\omega)$, although it is not dense!

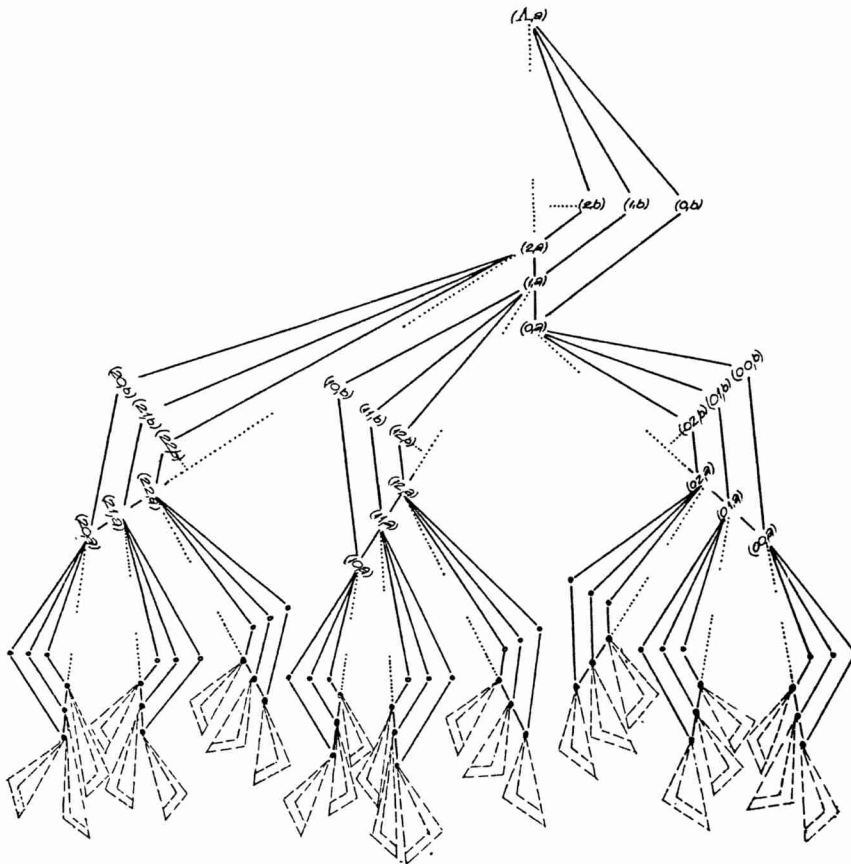


figure 2: domain E

My aim is to generalize the properties of i in order to get an exact characterization of all epis of $POS(Z)$, for arbitrary subset system Z . Note throughout the paper that the presence of a bottom \perp would not change any of the results. Thus every result holds for $POS_{\perp}(Z)$, too. Here I would like to thank J.Reiterman for helpful conversations on the topic of this paper.

2. A new closure operator

Notation: Ord denotes the class of all ordinals.

Definition 1: Let $P \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha \in \text{Ord}$ be arbitrary. Then "a is α -connected with b through X" - in symbols $a \stackrel{\alpha, X}{\vdash} b$ - if

1) For $\alpha = 0$

$\exists Y_b \subseteq P: b \in \text{cl}(Y_b) \ \& \ \forall y \in Y_b \ \exists x_y \in X: a \geq x_y \geq y$, and

2) For $\alpha > 0$

$\exists Y_b \subseteq P: b \in \text{cl}(Y_b) \ \& \ \forall y \in Y_b \ \exists b_y \in P \ \exists \alpha_y \in \text{Ord}:$

$y \leq b_y \ \& \ \alpha_y < \alpha \ \& \ a \stackrel{\alpha_y, X}{\vdash} b_y.$

Remark: Note that in the case of the counterexample in [3] for every $a \in (E \setminus B)$ $a \stackrel{0, B}{\vdash} a$ holds!

For the illustration of Definition 1 see the figure 3 at the end of this paragraph.

The following Lemmas 1-5 give some of the most important properties of the relation "to be connected through X" defined in Definition 1.

Lemma 1: For any $P \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha, \beta \in \text{Ord}$, if $\alpha < \beta$ then $a \stackrel{\alpha, X}{\vdash} b$ implies $a \stackrel{\beta, X}{\vdash} b$.

Proof: If $\beta = 0$ then evident.

Let $\beta > 0$. Then let $Y_b = \{b\}$. Since $b \in \text{cl}(Y_b)$ and $\forall y \in Y_b \ \exists \beta_y = \alpha < \beta \ \exists b_y = b \in P: b_y \geq y \ \& \ a \stackrel{\beta_y, X}{\vdash} b_y$, $a \stackrel{\beta, X}{\vdash} b$. \square

Lemma 2: For any $P \in |\text{POS}(Z)|$, $X, Y \subseteq P$, $\alpha \in \text{Ord}$ and $a, b \in P$, if $X \subseteq Y$ then $a \stackrel{\alpha, X}{\vdash} b$ implies $a \stackrel{\alpha, Y}{\vdash} b$.

Proof: by transfinite induction.

If $\alpha = 0$ then $a \perp_{\alpha, X} b$ means $\exists Y_b \subseteq P: b \in \text{cl}(Y_b)$ &
 $\forall y \in Y_b \exists x_y \in X \subseteq Y: a \geq x_y \geq y$. Hence $a \perp_{\alpha, Y} b$.
Let $\alpha > 0$ and suppose that $\forall \beta < \alpha$ the Lemma holds.
Then $a \perp_{\alpha, X} b$ means $\exists Y_b \subseteq P: b \in \text{cl}(Y_b)$ & $\forall y \in Y_b \exists b_y \in P$
 $\exists \alpha_y \in \text{Ord}: \alpha_y < \alpha$ & $b_y \geq y$ & $a \perp_{\alpha_y, X} b_y$. Since by the induc-
tion hypothesis $\forall y \in Y_b: a \perp_{\alpha_y, Y} b_y$, we get by Definition 1
 $a \perp_{\alpha, Y} b$. \square

Lemma 3: For any $P \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b, c \in P$ and $\alpha \in$
 Ord , if $a \geq b$ and $b \perp_{\alpha, X} c$ then $a \perp_{\alpha, X} c$.

Proof: by transfinite induction on α .

If $\alpha = 0$ then $b \perp_{\alpha, X} c$ means that $\exists Y_c \subseteq P: c \in \text{cl}(Y_c)$ &
 $\forall y \in Y_c \exists x_y \in X: b \geq x_y \geq y$. Since $a \geq b$, $\forall y \in Y_c: a \geq x_y \geq y$,
hence $a \perp_{\alpha, X} c$.

Let $\alpha > 0$ and suppose that $\forall \beta < \alpha$ the Lemma holds. Then
 $b \perp_{\alpha, X} c$ means that $\exists Y_c \subseteq P: c \in \text{cl}(Y_c)$ & $\forall y \in Y_c \exists \alpha_y \in \text{Ord}$
 $\exists b_y \in P: \alpha_y < \alpha$ & $b_y \geq y$ & $b \perp_{\alpha_y, X} b_y$. Then by induction hy-
pothesis $\forall y \in Y_c: a \perp_{\alpha_y, X} b_y$ hence by Definition 1 $a \perp_{\alpha, X} c$. \square

Lemma 4: For any $P \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b, c \in P$ and
 $\alpha \in \text{Ord}$, if $a \perp_{\alpha, X} b$ and $b \geq c$, then $a \perp_{\alpha+1, X} c$.

Proof: Let $Y_c = \{c\}$. Then $c \in \text{cl}(Y_c)$ & $\forall y \in Y_c$
 $\exists \alpha_y = \alpha \in \text{Ord} \exists b_y = b \in P: \alpha_y < \alpha + 1$ & $b_y \geq y$ &
 $a \perp_{\alpha_y, X} b_y$. Hence by Definition 1 $a \perp_{\alpha+1, X} c$. \square

Lemma 5: For any $P \in |\text{POS}(Z)|$, for any $X \subseteq P$ and for any
 $a \in \text{cl}(X): a \perp_{0, X} a$.

Proof: Let $Y_a := \{x \in X: a \geq x\}$. Then $a \in \text{cl}(Y_a)$ &
 $\forall y \in Y_a \exists x_y (=y) \in X: a \geq x_y \geq y$. Hence by Definition 1

$a \stackrel{0, X}{\sim} a$. \square

Remember that our aim essentially is to find the co-congruence relation of maps of $\text{POS}(Z)$. In the following we will see how the relation "To be connected through X " leads to this aim.

Proposition 1: For any $P \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b \in P$, $\alpha \in \text{Ord}$ and $\varphi, \psi: P \rightarrow Q \in \text{Mor POS}(Z)$, if $a \stackrel{\alpha, X}{\sim} b$ and $\varphi \upharpoonright X = \psi \upharpoonright X$, then $\varphi(a) \geq \psi(b)$ and $\psi(a) \geq \varphi(b)$.

Proof: by transfinite induction on α .

If $\alpha = 0$ then $a \stackrel{0, X}{\sim} b$ means $\exists Y_b \subseteq P: b \in \text{cl}(Y_b)$ & $\forall y \in Y_b \exists x_y \in X: a \geq x_y \geq y$. Then $\forall y \in Y_b \varphi(a) \geq \varphi(x_y) = \psi(x_y) \geq \psi(y)$ resp. $\psi(a) \geq \psi(x_y) = \varphi(x_y) \geq \varphi(y)$. Since $b \in \text{cl}(Y_b)$ implies $\psi(b) \in \text{cl}(\psi(Y_b))$ resp. $\varphi(b) \in \text{cl}(\varphi(Y_b))$ (since φ and ψ are Z -continuous), we get $\varphi(a) \geq \psi(b)$ resp. $\psi(a) \geq \varphi(b)$.

Let $\alpha > 0$ and suppose that $\forall \beta < \alpha$ the proposition is true.

Then $a \stackrel{\alpha, X}{\sim} b$ means that $\exists Y_b \subseteq P: b \in \text{cl}(Y_b)$ & $\forall y \in Y_b \exists \alpha_y \in \text{Ord} \exists b_y \in P \alpha_y < \alpha$ & $b_y \geq y$ & $a \stackrel{\alpha_y, X}{\sim} b_y$. By the induction hypothesis

$\forall y \in Y_b: \varphi(a) \geq \psi(b_y)$ resp. $\psi(a) \geq \varphi(b_y)$.

Then $\forall y \in Y_b: \varphi(a) \geq \psi(b_y) \geq \psi(y)$ resp. $\psi(a) \geq \varphi(b_y) \geq \varphi(y)$ and since $\psi(b) \in \text{cl}(\psi(Y_b))$ resp. $\varphi(b) \in \text{cl}(\varphi(Y_b))$, we get

$\varphi(a) \geq \psi(b)$ resp. $\psi(a) \geq \varphi(b)$. \square

Corollary 1: For any $P \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha \in \text{Ord}$, $a \stackrel{\alpha, X}{\sim} b$ implies $a \geq b$.

Proof: Let $\varphi = \psi = \text{id}_P$. \square

Corollary 2: For any $\underline{P} \in |\text{POS}(Z)|$, $X \subseteq P$, $a \in P$, $\alpha \in \text{Ord}$ and $\varphi, \psi: \underline{P} \rightarrow \underline{Q} \in \text{Mor POS}(Z)$, if $a \stackrel{\alpha, X}{\sim} a$ and $\varphi \upharpoonright X = \psi \upharpoonright X$ then $\varphi(a) = \psi(a)$!

Definition 2: Let $\underline{P} \in |\text{POS}(Z)|$ and $X \subseteq P$ be arbitrary. Then $\text{CL}(X) := \{a \in P \mid \exists \alpha \in \text{Ord}: a \stackrel{\alpha, X}{\sim} a\}$.

Having arrived at this point we know that $\text{CL}(X)$ is contained in the co-congruence relation of any map of $\text{POS}(Z)$ with image X (see Cor. 3 of Prop. 2).

Now let us prove some properties of $\text{CL}(X)$.

Lemma 6: For any $\underline{P} \in |\text{POS}(Z)|$ and $X \subseteq P$, $\text{cl}(X) \subseteq \text{CL}(X)$.

Proof: by Lemma 5. \square

Corollary: For any $\underline{P} \in |\text{POS}(Z)|$ and $X \subseteq P$, $X \subseteq \text{CL}(X)$.

Proposition 2: For any $\underline{P} \in |\text{POS}(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha \in \text{Ord}$, $a \stackrel{\alpha, \text{CL}(X)}{\sim} b \implies \exists \beta \in \text{Ord}: a \stackrel{\beta, X}{\sim} b$.

Proof: by transfinite induction on α .

Let $\alpha = 0$. Then $b \in \text{cl}(Y_b)$ & $\forall y \in Y_b \exists x_y \in \text{CL}(X): a \geq x_y \geq y$.

Since $x_y \in \text{CL}(X) \exists \alpha_y \in \text{Ord}: x_y \stackrel{\alpha_y, X}{\sim} x_y$. Then by Lemma 3 $a \stackrel{\alpha_y, X}{\sim} x_y$. Now $b \in \text{cl}(Y_b)$ & $\forall y \in Y_b \exists x_y \in P \exists \alpha_y \in \text{Ord}$:

$\alpha_y < \beta := \sum_{y \in Y_b} (\alpha_y + 1)$ & $x_y \geq y$ & $a \stackrel{\alpha_y, X}{\sim} x_y$. Then by Definition 1 $a \stackrel{\beta, X}{\sim} b$.

Let $\alpha > 0$ and suppose that for any $\delta < \alpha$ the proposition holds. Then $a \stackrel{\alpha, \text{CL}(X)}{\sim} b$ means that $\exists Y_b \subseteq P: b \in \text{cl}(Y_b)$ & $\forall y \in Y_b \exists b_y \in P \exists \alpha_y \in \text{Ord}: \alpha_y < \alpha$ & $b_y \geq y$ & $a \stackrel{\alpha_y, \text{CL}(X)}{\sim} b_y$ but then by induction hypothesis $\forall y \in Y_b \exists \beta_y \in \text{Ord}: a \stackrel{\beta_y, X}{\sim} b_y$.

Then by Definition 1 for $\beta := \sum_{y \in Y_b} (\beta_y + 1)$ $a \vdash_{\beta} X$ b. \square

Corollary 1: For any $\underline{P} \in |\text{POS}(Z)|$ and $X \subseteq P: \text{CL}(\text{CL}(X)) = \text{CL}(X)$.

Proof: by Corollary of Lemma 6 and by Proposition 2 for $a = b$. \square

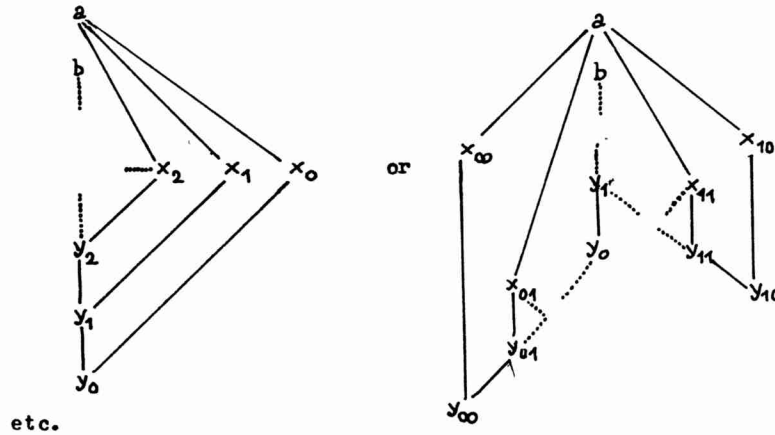
Corollary 2: For any $\underline{P} \in |\text{POS}(Z)|$ the operator $\text{CL}: \mathcal{P}(P) \rightarrow \mathcal{P}(P)$, which assigns to each $X \subseteq P$ $\text{CL}(X)$, is a closure operator.

Proof: By Corollary of Lemma 6 $X \subseteq \text{CL}(X)$, by Lemma 2 if $X \subseteq Y$ then $\text{CL}(X) \subseteq \text{CL}(Y)$ and by the above Corollary 1 $\text{CL}(\text{CL}(X)) = \text{CL}(X)$. \square

Corollary 3: For any $\underline{P} \in |\text{POS}(Z)|$, $X \subseteq P$ and $\varphi, \psi: \underline{P} \rightarrow \underline{Q} \in \text{Mor POS}(Z)$, if $\varphi \upharpoonright X = \psi \upharpoonright X$ then $\varphi \upharpoonright \text{CL}(X) = \psi \upharpoonright \text{CL}(X)$.

Proof: by Corollary 2 of Proposition 1. \square

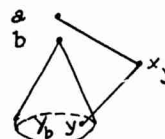
For $\alpha = 0$ and for $Z = \omega$



Further on we shall symbolize $b \in \text{cl}(Y_b)$ by



i.e. for $\alpha = 0$ the figure looks like that:



For $\alpha > 0$ we illustrate then Definition 1 by:

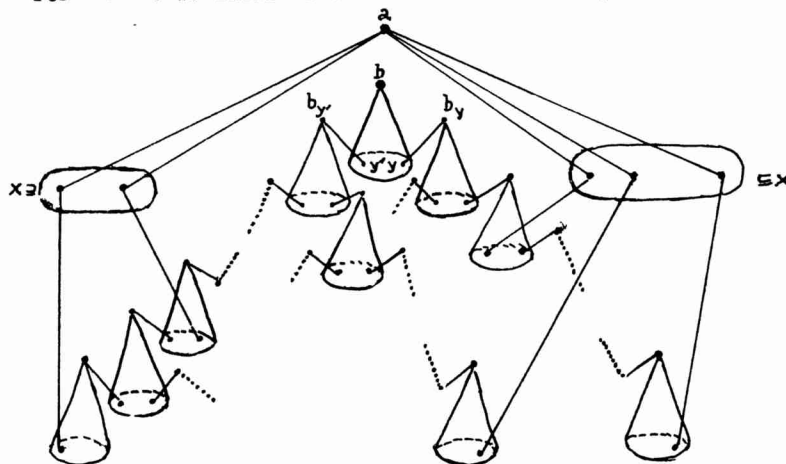


figure 3: Illustration of Definition 1.

3. The epis of $\text{POS}(Z)$. Now we are ready to give the exact characterization of epis in $\text{POS}(Z)$:

Theorem: Any $f: P \rightarrow Q \in \text{Mor } \text{POS}(Z)$ is an epi iff $\text{CL}(f(P)) = Q$.

Proof: If $\text{CL}(f(P)) = Q$ then f epi follows immediately from Corollary 3 of Proposition 2.

Let $f: P \rightarrow Q \in \text{Mor } \text{POS}(Z)$ be an epi in $\text{POS}(Z)$. Denote

$CL(f(P)) = Q_0$ and suppose $Q - Q_0 \neq \emptyset$. Then we construct $\hat{\varphi}, \hat{\psi}: Q \rightarrow \hat{R} \in \text{Mor POS}(Z)$ with $\hat{\varphi} \cdot f = \hat{\psi} \cdot f$ but $\hat{\varphi} \neq \hat{\psi}$ which contradicts the epiness of f . First we define the maps φ and ψ , namely $\varphi := \text{id}_Q$ and $\psi := \text{id}_{Q_0} \cup \wp$, where $\wp: Q - Q_0 \hookrightarrow \text{im } \wp$ is a bijection ($Q - Q_0 \neq \emptyset$) with $\text{im } \wp \cap (Q - Q_0) = \emptyset$. Let the set $R := Q \cup \text{im } \wp$.

We are going to define on R the relation \leq_R : Let $a, b \in R$, then

$$a \leq_R b \text{ iff } \begin{cases} \text{A: } a \leq_Q b \text{ if } a, b \in Q, \\ \text{B: } \wp^{-1}(a) \leq_Q \wp^{-1}(b) \text{ if } a, b \in \text{im } \wp, \\ \text{C: } \wp^{-1}(b) \stackrel{\alpha, Q_0}{\vdash} a \text{ for an } \alpha \in \text{Ord} \text{ if } a \in Q, b \in \text{im } \wp, \\ \text{D: } b \stackrel{\alpha, Q_0}{\vdash} \wp^{-1}(a) \text{ for an } \alpha \in \text{Ord} \text{ if } a \in \text{im } \wp, b \in Q. \end{cases}$$

In the following we shall prove that \leq_R is a partial order.

- 1) reflexive: $\forall a \in R: a \in Q \Rightarrow a \leq_Q a \xrightarrow{A} a \leq_R a$ and $a \in \text{im } \wp \Rightarrow \wp^{-1}(a) \leq_Q \wp^{-1}(a) \xrightarrow{B} a \leq_R a$.
- 2) antisymmetric: let $a \leq_R b$ and $b \leq_R a$. Then
 - a) $a, b \in Q \xrightarrow{A} a = b$
 - b) $a, b \in \text{im } \wp \xrightarrow{B} \wp^{-1}(a) = \wp^{-1}(b) \Rightarrow a = b$
 - c) $a \in Q, b \in \text{im } \wp \xrightarrow{C, D} \wp^{-1}(b) \stackrel{\alpha, Q_0}{\vdash} a$ and $a \stackrel{\alpha, Q_0}{\vdash} \wp^{-1}(b) \xrightarrow{\text{Cor. 1 of Prop. 1}} \wp^{-1}(b) \geq a$ and $a \geq \wp^{-1}(b) \Rightarrow \wp^{-1}(b) = a$

$\xrightarrow{\text{Def. of } \wp} a \in Q - Q_0$. But $a = \wp^{-1}(b) \stackrel{\alpha, Q_0}{\vdash} a$ implies $a \in CL(Q_0) = Q_0$ which is a contradiction. This means:

$$a \in Q, b \in \text{im } \wp \Rightarrow \neg(a \leq_R b \ \& \ b \leq_R a).$$

d) For $a \in \text{im } \wp, b \in Q$ we get the same.

3) transitive: let $a \leq_R b \leq_R c$. Then:

$$\text{a) } a, b, c \in Q \xrightarrow{A} a \leq_R c.$$

- b) $a, b, c \in \text{im } \varphi \xrightarrow{B} a \leq_R c$
- c) $a \in Q, bc \in \text{im } \varphi \xrightarrow{C, B} \varphi^{-1}(b) \vdash_{\alpha, Q_0} a$ and
 $\varphi^{-1}(b) \leq_Q \varphi^{-1}(c) \xrightarrow{\text{Lemma 3}} \varphi^{-1}(c) \vdash_{\alpha, Q_0} a \xrightarrow{C} a \leq_R c.$
- d) For $a \in \text{im } \varphi, b, c \in Q$ dually by D and A.
- e) $a, c \in \text{im } \varphi, b \in Q \xrightarrow{D, C} b \vdash_{\alpha, Q_0} \varphi^{-1}(a)$ and
 $\varphi^{-1}(c) \vdash_{\alpha, Q_0} b \xrightarrow{\text{Cor. 1 of Prop. 1}} \varphi^{-1}(a) \leq_Q b \leq_Q \varphi^{-1}(c) \xrightarrow{B} a \leq_R c.$
- f) For $a, c \in Q, b \in \text{im } \varphi$ dually by C, D and A.
- g) $a, b, c \in Q, c \in \text{im } \varphi \xrightarrow{A, C} a \leq_Q b$ and
 $\varphi^{-1}(c) \vdash_{\alpha, Q_0} b \xrightarrow{\text{Lemma 4}} \varphi^{-1}(c) \vdash_{\alpha+1, Q_0} a \xrightarrow{C} a \leq_R c.$
- h) For $a, b \in \text{im } \varphi, c \in Q$ dually by B and D.

Next we shall prove that φ and ψ are Z-continuous. For this let Y be an arbitrary Z-set in Q with $a = \sup_R Y$. First we show that $\varphi(a) = \sup_R \varphi(Y)$. By A (of the definition of \leq_R) it is clear that $a \geq_R y \forall y \in Y$ and that $\forall b \in Q$, if $y \leq_R b \forall y \in Y$, then also $a \leq_R b$ holds. Let $b \in \text{im } \varphi$ with $y \leq_R b \forall y \in Y$. Then by C $\varphi^{-1}(b) \vdash_{\alpha, Q_0} y \forall y \in Y$, which by Definition 1 implies $\varphi^{-1}(b) \vdash_{\alpha, Q_0} a$, where $\alpha := \sum_{y \in Y} (\alpha_y + 1)$. Again by C we get $a \leq_R b$. Now we are going to prove that $\psi(a) = \sup_R \psi(Y)$.

1. Suppose $a \in Q_0$ (and hence $\psi(a) = a$). By A $y \leq_R a \forall y \in Y \cap Q_0$ and by D $\varphi(y) \leq_R a \forall y \in Y \cap (Q - Q_0)$ since for these $y - a \vdash_{0, Q_0} y$. Hence $z \leq_R a \forall z \in \psi(Y)$. Let $z \leq_R b \forall z \in \psi(Y)$.
- a) if $b \in Q$ then by A $z \leq_Q b \forall z \in \psi(Y) \cap Q_0$ and by D and Corollary 1 of Proposition 1 $\varphi^{-1}(z) \leq_Q b \forall z \in \psi(Y) \cap \text{im } \varphi$. This means $y \leq_Q b \forall y \in Y$, hence $a \leq_Q b$ and hence by A $a \leq_R b$.

b) Suppose $b \in \text{im } \varphi$. Then by B $\varphi^{-1}(z) \leq_Q \varphi^{-1}(b)$
 $\forall z \in \psi(Y) \cap \text{im } \varphi$ and by C and Corollary 1 of Proposition 1
 $z \leq_Q \varphi^{-1}(b) \forall z \in \psi(Y) \cap Q_0$, i.e. $z \leq_Q \varphi^{-1}(b) \forall y \in Y$. This im-
 plies $a \leq_Q \varphi^{-1}(b)$. Since $a \in Q_0$ we can write $\varphi^{-1}(b) \vdash_{\underline{0}, Q_0} a$,
 hence by C $a \leq_R b$.

2. Let $a \in Q - Q_0$ (and hence $\psi(a) = \varphi(a)$). By B
 $z \leq_R \varphi(a) \forall z \in \psi(Y) \cap \text{im } \varphi$ and by C also $z \leq_R \varphi(a) \forall z \in \psi(Y) \cap$
 $\cap Q_0$ since for these z 's $a \vdash_{\underline{0}, Q_0} z$. Now let $z \leq_R b \forall z \in \psi(Y)$

a) If $b \in Q$ then by D $b \vdash_{\underline{\alpha_z}, Q_0} \varphi^{-1}(z) \forall z \in \psi(Y) \cap$
 $\cap \text{im } \varphi$ and by A and the above remark $b \vdash_{\underline{0}, Q_0} z \forall z \in \psi(Y) \cap$
 $\cap Q_0$. Applying Definition 1 we get $b \vdash_{\underline{\alpha}, Q_0} a$, where
 $\alpha := \sum_{z \in \psi(Y) \cap \text{im } \varphi} (\alpha_z + 1)$, which by D means $\varphi(a) \leq_R b$.

b) If $b \in \text{im } \varphi$ then by B $\varphi^{-1}(z) \leq_Q \varphi^{-1}(b)$
 $\forall z \in \psi(Y) \cap \text{im } \varphi$ and by C and Corollary 1 of Proposition 1
 $y \leq_Q \varphi^{-1}(b) \forall y \in Y \cap Q_0$, i.e. $y \leq_Q \varphi^{-1}(b) \forall y \in Y$, hence
 $a \leq_Q \varphi^{-1}(b)$. By B this means $\varphi(a) \leq_R b$.

By Banaschewski-Nelson [2] or Meseguer [6] $\text{POS}(Z)$ is (full-mono)-
 reflective in $Z\text{POS}$ - the category of posets and Z -continuous maps.
 Let η_R denote the (Z -continuous) $\text{POS}(Z)$ -reflection of $\underline{R} := (R, \leq_R)$
 and let $\hat{\psi} := \eta_R \cdot \psi$ and $\hat{\varphi} := \eta_R \cdot \varphi$. Then $\hat{\psi} \neq \hat{\varphi}$ since $\varphi \neq \psi$
 and $\hat{\psi} \cdot f = \hat{\varphi} \cdot f$ since $\psi \cdot f = \varphi \cdot f$. \square

4. Some consequences

Corollary 1: An $m: \underline{P} \rightarrow \underline{Q} \in \text{Mor } \text{POS}(Z)$ is an extremal mono
 iff it is full (i.e. $m(a) \leq_Q m(b)$ iff $a \leq_P b \forall a, b \in P$) and
 $\text{CL}(m(P)) = m(P)$.

Proof: 1) Let m be an extremal mono and let $m=f \cdot e$, with $e:\underline{P} \rightarrow \underline{R}$, $R := CL(m(P))$, $\leq_R := \leq_Q \cap R^2$ and $e:=m$ and $f:\underline{R} \rightarrow \underline{Q}$, $f:=id_R$. Since by Lemma 6 $cl(R) = cl(CL(m(P))) = CL(m(P)) = R$, $\underline{R} \in |POS(Z)|$, thus $e, f \in Mor\ POS(Z)$. By the Theorem e is an epi, hence e is an isomorphism. This implies $m(P)=e(P)=R=CL(m(P))$ and $m^2(\leq_P) = e^2(\leq_P) = \leq_R = \leq_Q \cap R^2 = \leq_Q \cap \cap(m(P))^2$.

2) Let m be full and let $CL(m(P)) = m(P)$. Suppose $m = f \cdot e$ with $e:\underline{P} \rightarrow \underline{R}$, $f:\underline{R} \rightarrow \underline{Q} \in Mor\ POS(Z)$ and e epi. We have to show that e is an isomorphism.

By the Theorem we know that $CL(e(P)) = R$. Then $f(R) = f(CL(e(P)))$. By the Corollary of the following Lemma 7 $f(CL(e(P))) \subseteq CL(f(e(P))) = CL(m(P)) = m(P)$.

Before going on let us prove the

Lemma 7: For any $f:\underline{P} \rightarrow \underline{Q} \in Mor\ POS(Z)$, $X \subseteq P$, $\alpha \in Ord$ and $a, b \in P$, if $a \vdash_{\alpha, X} b$ then $f(a) \vdash_{\alpha, f(X)} f(b)$.

Proof: By transfinite induction.

For $\alpha = 0 \exists Y_b \subseteq P: b \in cl(Y_b) \ \& \ \forall y \in Y_b \exists x_y \in X: a \geq x_y \geq y$ hence $f(a) \geq f(x_y) \geq f(y)$. Since $f(b) \in cl(f(Y_b))$ we get by Definition 1 $f(a) \vdash_{0, f(X)} f(b)$.

Let $\alpha > 0$ and suppose that for any $\beta < \alpha$ the Lemma holds.

Then $a \vdash_{\alpha, X} b$ means that $\exists Y_b \subseteq P: b \in cl(Y_b) \ \& \ \forall y \in Y_b \exists b_y \in P \exists \alpha_y \in Ord: \alpha_y < \alpha \ \& \ b_y \geq y \ \& \ a \vdash_{\alpha_y, X} b_y$ hence

$f(b_y) \geq f(y) \ \& \ f(a) \vdash_{\alpha_y, f(X)} f(b_y)$. Again, since $f(b) \in cl(f(Y_b))$, we then get $f(a) \vdash_{\alpha, f(X)} f(b)$.

□ Lemma 7.

Corollary: For any $f: P \rightarrow Q \in \text{Mor POS}(Z)$ and $X \subseteq P$,
 $f(\text{CL}(X)) \subseteq \text{CL}(f(X))$.

Let us continue the proof. We have got that $f(\text{CL}(e(P))) \subseteq m(P)$.
Then if we knew that $e \cdot m^{-1} \cdot f \in \text{Mor POS}(Z)$ then we would get
that $\text{CL}(e(P)) = e(P)$, i.e. that e is surjective, since
 $e \cdot m^{-1} \cdot f = \text{id}_R$ (since e is epi and $e \cdot m^{-1} \cdot f \cdot e = e \cdot m^{-1} \cdot m = e \cdot \text{id}_P =$
 $= \text{id}_R \cdot e$) and so $\forall a \in \text{CL}(e(P)) \quad e(m^{-1}(f(a))) = a$, i.e. $a \in e(P)$.
But even $m^{-1} \cdot f \in \text{Mor POS}(Z)$ since for any Z -set $A \subseteq R$ with
 $a = \sup(A)$, $f(A)$ is a Z -set and $f(a) = \sup_{\leq Q} f(A)$ and since
 $f(R) \subseteq m(P)$ and m is full $m^{-1}(f(A))$ is a Z -set in P , so it
must have a supremum and this is $m^{-1}(f(a))$. Since $\forall a, b \in P$
 $e(a) \leq e(b) \implies f(e(a)) \leq f(e(b)) \implies m(a) \leq m(b) \implies a \leq b$ e is
also full. Thus e is full and surjective, i.e. an isomorphism
and hence m is an extremal mono.

□ Corollary 1.

Corollary 2: $\text{POS}(Z)$ is co-(well-powered).

Proof: It is enough to prove the following

Lemma 8: For any $P \in |\text{POS}(Z)|$ and $X \subseteq P$, $\text{CL}(X) \subseteq \bar{X}$, where
 \bar{X} is the join-closure of X in P , i.e. $\bar{X} = \{\sup S : S \subseteq X\}$.

Proof: We are going to prove by transfinite induction
on α that $a \uparrow_{\alpha, X} a$ implies $a = \sup X_a$, where $X_a = \{x \in X : a \geq x\}$.
Therefore let $b \geq x \quad \forall x \in X_a$ for some $b \in P$. If $\alpha = 0$ then
 $\exists Y_a \subseteq P : a \in \text{cl}(Y_a) \ \& \ \forall y \in Y_a \ \exists x_y \in X : a \geq x_y \geq y$, which immedi-
ately implies $b \geq a$.

Now suppose $\alpha > 0$ and that for any $\beta < \alpha$ the statement holds.
Then $\exists Y_a \subseteq P : a \in \text{cl}(Y_a) \ \& \ \forall y \in Y_a \ \exists \alpha_y < \alpha \ \exists b_y \in P : b_y \geq y \ \&$
 $\& \ a \uparrow_{\alpha_y, X} b_y$. Since by Cor. 1 of Prop. 1 $\forall y \in Y_a \quad X_{b_y} \subseteq X_a$,

by induction hypothesis we get that $\forall y \in Y_a \quad b \geq b_y$. Then $b_y \geq y$ and $a \in \text{cl}(Y_a)$ imply $b \geq a$.

□ Lemma 8 and Corollary 2

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