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THE NUMBER OF MINIMAL VARIETIES OF IDEMPOTENT GROUPOIDS

Jaroslav JEŽEK

Abstract: It is proved that there are uncountably many minimal varieties of commutative idempotent groupoids.

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Kalicki [2] proved that there are uncountably many minimal varieties of commutative groupoids. Although this result was strengthened and generalized in various ways (see e.g. [1],[3],[4],[5]), there seems to be no mention of idempotency in the literature in this connection. The purpose of this paper is to prove the following

Theorem. There are $2^{\aleph_0}$ minimal varieties of commutative idempotent groupoids.

The proof will be divided into several lemmas. It will be convenient to work in the free commutative groupoid G over $\{x,y\}$ (x,y are two different elements). The binary operation of G will be denoted multiplicatively. If $a,b,c,d \in G$ then $ab = cd$ takes place iff either $a=c$ & $b=d$ or $a=d$ & $b=c$. G is a cancellation groupoid. There exists a unique mapping λ of G into the set of positive integers such that $\lambda(x) = \lambda(y) = 1$

and $\lambda(ab) = \lambda(a) + \lambda(b)$ for all $a,b \in G$; the number $\lambda(a)$ is called the length of an element $a \in G$. An element $a \in G$ is said to be a subterm of an element $b \in G$ if $b = (((ac_1)c_2\dots \dots)c_k$ for some $k \geq 0$ and some elements $c_1, c_2, \dots, c_k \in G$; if $k \geq 1$, a is said to be a proper subterm of b. Evidently, an element $a \in G$ is a proper subterm of $b_1 b_2$ iff it is a subterm of either b_1 or b_2.

If $n \geq 0$ and $a,b \in G$, we define an element $[a,b]^n \in G$ as follows: $[a,b]^0 = a$; $[a,b]^{n+1} = [a,b]^n b$. Hence $[a,b]^n = (((ab)b)\dots)b$ with n appearances of b.

Put $N = \{2,3,4,\dots\}$. Denote by E the set of all finite sequences $(e_1,\dots,e_k)$ such that $k \geq 1$, $e_1 \in N$ and $e_i \in N \times \{1,2\}$ for all $i \in \{2,\dots,k\}$.

In the following let M be an arbitrary subset of N.

For every $e \in E$ define three elements R_e, S_e, T_e of G as follows:

(1) Let $e=(n)$, $n \in N$. Then $R_e = [x,y]^n x$, $S_e = [x,y]^{2n} x$, $T_e = x$ if $n \in M$ and $T_e = y$ if $n \notin M$.

(2) Let $e=(f,(n,1))$, $f \in E$, $n \in N$. Then $R_e = [T_f, S_f]^{n-1} R_f$, $S_e = [T_f, S_f]^{2n-1} R_f$, $T_e = R_f$ if $n \in M$ and $T_e = S_f$ if $n \notin M$.

(3) Let $e=(f,(n,2))$, $f \in E$, $n \in N$. Then $R_e = [T_f, R_f]^{n-1} S_f$, $S_e = [T_f, R_f]^{2n-1} S_f$, $T_e = S_f$ if $n \in M$ and $T_e = R_f$ if $n \notin M$.

Lemma 1. Let $e \in E$ and let p be an endomorphism of G. Then $p(R_e)$ is shorter than $p(S_e)$; $p(T_e)$ is a proper subterm of both $p(R_e)$ and $p(S_e)$.

Proof. It is obvious.

Lemma 2. Let $n,m \geq 2$ and let $a,b,c,d \in G$ be such that $[a,b]^{n-1} = [c,d]^{m-1}$ and $[a,b]^{2n-1} = [c,d]^{2m-1}$. Then $n=m$, $a=c$ and

$b=d$.

Proof. It is enough to consider the case $n \leq m$. We have $b=d$, since otherwise $b=[c,d]^{m-2}=[c,d]^{2m-2}$, which is impossible. From this we get by cancellation $a=[c,b]^{m-n}$ and $a= =[c,b]^{2m-2n}$; hence $m-n=2m-2n$, i.e. $m=n$; we get $a=c$ as a consequence.

Lemma 3. Let $e,f \in E$ and let p,q be two endomorphisms of G such that $p(R_eS_e)=q(R_fS_f)$. Then $e=f$ and $p=q$.

Proof. By induction on the sum of the lengths of e and f. If e,f are both one-termed, it is evident. Suppose $e=(m)$ and $f=(g,(n,1))$. We have $p([x,y]^m x)=q([T_g,S_g]^{n-1}R_g)$ and $p([x,y]^{2m}x)=q([T_g,S_g]^{2n-1}R_g)$. Evidently $p(x)=q(R_g)$, $p([xy,y]^{m-1})=q([T_g,S_g]^{n-1})$ and $p([xy,y]^{2m-1})=q([T_g,S_g]^{2n-1})$. By Lemma 2 we get $n=m$ and $p(xy)=q(T_g)$, so that $q(T_g)$ is longer than $p(x)=q(R_g)$, which is impossible by Lemma 1. Quite similarly, we cannot have $e=(m)$ and $f=(g,(n,2))$.

Let $e=(g,(n,1))$ and $f=(j,(m,1))$. We have $p([T_g,S_g]^{n-1}R_g)= q([T_h,S_h]^{m-1}R_h)$ and $p([T_g,S_g]^{2n-1}R_g)=q([T_h,S_h]^{2m-1}R_h)$. Evidently $p(R_g)=q(R_h)$, $p([T_g,S_g]^{n-1})=q([T_h,S_h]^{m-1})$ and $p([T_g,S_g]^{2n-1})=q([T_h,S_h]^{2m-1})$. By Lemma 2, $n=m$ and $p(S_g) = =q(S_h)$. By the induction assumption, $g=h$ and $p=q$; since $n=m$, we get $e=f$.

If $e=(g,(n,2))$ and $f=(h,(m,2))$, the proof is quite analogous.

Suppose $e=(g,(n,1))$ and $f=(h,(m,2))$. Similarly as above we get $p(R_g)=q(S_h)$ and $p(S_g)=q(R_h)$. However, this is a contradiction by Lemma 1.

Denote by A the set of all $a \in G$ such that whenever $e \in E$ and p is an endomorphism of G then neither p(xx) nor $p(R_e S_e)$ is a subterm of a. Define a binary operation $\circ$ on A as follows:

(1) if $a,b \in A$ and $ab \in A$, put $a \circ b = ab$;

(2) if $a \in A$, put $a \circ a = a$;

(3) if $a,b \in A$ and $ab = p(R_e S_e)$ for some $e \in E$ and some endomorphism p of G, put $a \circ b = p(T_e)$.

The correctness of this definition follows from Lemmas 1 and 3. Evidently $A(\circ)$ is a commutative idempotent groupoid.

Lemma 4. Let $a,b \in A$ and $ab \notin A$. Then either a=b or there are elements $R,S,T \in G$ with $R \neq S$ and a number $m \geq 2$ such that $ab = ([T,S]^{m-1}R)([T,S]^{2m-1}R)$.

Proof. It is easy.

Lemma 5. Let $u,v \in A$ and let u be a proper subterm of v. Then $uv \in A$.

Proof. There are an integer $k \geq 1$ and elements $w_1, \dots$ $\dots, w_k \in G$ with $v = (((uw_1)w_2)\dots)w_k$. Suppose $uv \notin A$. It follows from Lemma 4 that we can write $u = [T,S]^{m-1}R$ and $v = [T,S]^{2m-1}R$ for some R,S,T,m with $R \neq S$ and $m \geq 2$. Let us prove by induction on $j=1,\dots,k$ that $2m-j > 0$ and $(((uw_1)w_2)\dots)w_{k-j} = [T,S]^{2m-j}$. For j=1 it follows from $(((uw_1)w_2)\dots)w_k = [T,S]^{2m-1}R$, since we cannot have $(((uw_1)w_2)\dots)w_{k-1} = R$. Assume that the two assertions are proved for some $j < k$. If it were $(((uw_1)w_2)\dots$ $\dots)w_{k-j-1} = S$ then we would have $\lambda(u) \leq \lambda(S)$; but u is longer than S, a contradiction. Thus there remains only one possibility: $(((uw_1)w_2)\dots)w_{k-j-1} = [T,S]^{2m-j-1}$. If it were $2m-j-1=0$

then we would have $\lambda(u) \leq \lambda(T)$; but u is longer than T, a contradiction. Hence $2m-j-1>0$. The induction is thus finished. Especially, for j=k we get: there is an $i>0$ with $u = [T,S]^i$. Hence $[T,S]^{m-1}R = [T,S]^i$. We cannot have $S=[T,S]^{m-1}$ and so we get S=R, a contradiction.

<u>Lemma 6</u>. Let $a,b \in A$, $ab \notin A$ and $a \neq b$; let $i \geq 1$. Then $[a \circ b,b]^i a \in A$.

Proof. Since $a \circ b$ is a proper subterm of b, several applications of Lemma 5 give $[a \circ b,b]^i \in A$. Suppose $[a \circ b,b]^i a \notin A$. If it were $[a \circ b,b]^i = a$ then b would be a proper subterm of a, so that $ab \in A$ by Lemma 5, a contradiction. By Lemma 4 we get $[a \circ b,b]^i a = ([T,S]^{m-1}R)([T,S]^{2m-1}R)$ for some R,S,T,m with $R \neq S$ and $m \geq 2$. If it were $[a \circ b,b]^i = [T,S]^{m-1}R$ and $a = [T,S]^{2m-1}R$, then we would have either b=R or $b=[T,S]^{m-1}$, so that b would be a proper subterm of a and so $ab \in A$ by Lemma 5, a contradiction. Hence $[a \circ b,b]^i = [T,S]^{2m-1}R$ and $a = [T,S]^{m-1}R$. Since b=R is impossible, we get $b=[T,S]^{2m-1}$. By Lemma 4 there are r,s, $t \in G$ and a $k \geq 2$ such that $ab = ([t,s]^{k-1}r)([t,s]^{2k-1}r)$. There are two possible cases.

Case 1: $a = [T,S]^{m-1}R = [t,s]^{k-1}r$ and $b = [T,S]^{2m-1} = [t,s]^{2k-1}r$. Since either $r = [T,S]^{2m-2}$ or r=S, we cannot have $r = [T,S]^{m-1}$. Hence r=R. Since $R \neq S$, we get $[t,s]^{2k-1} = S$ and $[t,s]^{k-1} = [T,S]^{m-1}$, evidently a contradiction.

Case 2: $a = [T,S]^{m-1}R = [t,s]^{2k-1}r$ and $b = [T,S]^{2m-1} = [t,s]^{k-1}r$. Similarly as in the previous case we get $[t,s]^{k-1} = S$ and $[t,s]^{2k-1} = [T,S]^{m-1}$; we have either S=s or $S = [t,s]^{2k-2}$, evidently a contradiction.

Lemma 7. Let $n \in N$. Then the groupoid $A(\circ)$ satisfies the identity $R_{(n)}S_{(n)}=T_{(n)}$.

Proof. Let φ be any homomorphism of G into $A(\circ)$; we must prove $\varphi(R_{(n)}S_{(n)})=\varphi(T_{(n)})$. Put $a=\varphi(x)$ and $b=\varphi(y)$. If $a=b$, everything is clear. If $ab \in A$ then by Lemma 5, $\varphi(R_{(n)}S_{(n)})=[a,b]^n a \circ [a,b]^{2n} a=\varphi(T_{(n)})$. It remains to consider the case when $ab=p(R_e S_e)$ for some $e \in E$ and some endomorphism p of G; we have $a \circ b=p(T_e)$. There are two possible cases.

Case 1: $a=p(R_e)$ and $b=p(S_e)$. By Lemma 6 we have $\varphi(R_{(n)}S_{(n)})=[a \circ b,b]^{n-1}a \circ [a \circ b,b]^{2n-1}a=p(R_{(e,(n,1))}) \circ$ $\circ\, p(S_{(e,(n,1))})=p(T_{(e,(n,1))})=\varphi(T_{(n)})$.

Case 2: $a=p(S_e)$ and $b=p(R_e)$. By Lemma 6 we have $\varphi(R_{(n)}S_{(n)})=[a \circ b,b]^{n-1}a \circ [a \circ b,b]^{2n-1}a=p(R_{(e,(n,2))}) \circ$ $\circ\, p(S_{(e,(n,2))})=p(T_{(e,(n,2))})=\varphi(T_{(n)})$.

The proof of the Theorem can now be completed in the following way. For any subset M of N denote by V_M the variety of commutative idempotent groupoids determined by the identities $([x,y]^n x)([x,y]^{2n}x)=x$ for any $n \in M$ and $([x,y]^n x)([x,y]^{2n}x)=y$ for any $n \in N \setminus M$. It follows from Lemma 7 that V_M is non-trivial, so that it contains a minimal subvariety U_M. If M_1, M_2 are two different subsets of N, then evidently $V_{M_1} \cap V_{M_2}$ is trivial and so $U_{M_1} \neq U_{M_2}$. Hence the number of minimal varieties of commutative idempotent groupoids cannot be smaller than the number of subsets of N, i.e. than $2^{\aleph_0}$. On the other hand, it cannot be larger than $2^{\aleph_0}$, since there are only $2^{\aleph_0}$ varieties of groupoids.

R e f e r e n c e s

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Matematicko-fyzikální fakulta, Universita Karlova, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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