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TWO-VALUED MEASURE NEED NOT BE PURELY $\aleph_0$-COMPACT

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Abstract: The conjecture of Z. Frolík and J. Pachl ([2]) stated in the title is true (purely $\aleph_0$-compact measures were introduced in [2]).

Key words: Purely $\aleph_0$-compact measure.

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This note is closely related to the paper "Pure measures" by Z. Frolík and J. Pachl ([2]). We answer in the affirmative the conjecture stated there [2, 4,2(c)] and in the title of this note. For the definition of a purely $\aleph_0$-compact measure see the above mentioned paper. Our measure will be defined on a special σ-algebra, we call it $\mathcal{B}(I)$, and we will describe it now.

Let I be any index set. For $J \subseteq I$, p_J denotes a canonical projection of $\{0,1\}^I$ onto $\{0,1\}^J$. $\mathcal{A}$ denotes the σ-algebra generated by the family of sets $\{p^{-1}_{\{i\}}(1) : i \in I\}$. Let $X(J) \subseteq \subseteq \{0,1\}^J$ be the set of points all but finitely many coordinates of which are zero. Put $\mathcal{B}(I) = \{A \cap X(I) : A \in \mathcal{A}\}$.

The following properties of $\mathcal{B}(I)$ are easily established. For any set $B \in \mathcal{B}(I)$ there are a countable set $J(B) \subseteq I$ and a set $B \subseteq X(J(B))$ such that $B = p^{-1}_{J(B)}(B) \cap X(I)$. If two points $x, y \in X(I)$ are different only on coordinates not in $J(B)$ then either $\{x,y\} \subseteq B$, or $\{x,y\} \cap B = \emptyset$.

Two further properties of $\mathcal{B}(I)$ are a little less obvious.

(i) Any σ-algebra generated by a countable subfamily of $\mathcal{B}(I)$ has countable many atoms.

(ii) $\mathcal{B}(I)$ satisfies the continuum chain condition (i.e. any family $\mathcal{F} \subseteq \mathcal{B}(I)$ of nonempty pairwise disjoint sets has cardinality at most continuum - the cardinality of the real line).

Proof. (i) Let $\mathcal{C} \subseteq \mathcal{B}(I)$ be the smallest σ-algebra containing a family $\{C_1, C_2, \dots\} \subseteq \mathcal{B}(I)$. Let $A_i = p_{\{i\}}^{-1}(1)$, and $\mathcal{D}$ be a σ-subalgebra of $\mathcal{A}$ generated by a family $\{A_i : i \in J\}$, where $J = J(C_1) \cup J(C_2) \cup \dots$. J is countable. Any atom of $\mathcal{D}$ is of the form

$$\bigcap\{A_i : i \in K\} \cap \bigcap\{\{0,1\}^I - A_i : i \in J - K\},$$

for some $K \subseteq J$. Only countably many of these are not disjoint with $X(I)$ (those with K finite), so the σ-algebra $\mathcal{D} \cap X(I) = \{D \cap X(I) : D \in \mathcal{D}\}$ on $X(I)$ has only countably many atoms. $\mathcal{C}$ is a σ-subalgebra of $\mathcal{D} \cap X(I)$, then it has only countably many atoms, too.

(ii) Let $\mathcal{F} \subseteq \mathcal{B}(I)$ be a family of nonempty pairwise disjoint sets. For any $B \in \mathcal{F}$ take the set $A(B) = p_{J(B)}^{-1}(p_{J(B)}(B))$. $A(B)$ belongs to $\mathcal{A}$ and $\mathcal{F}' = \{A(B) : B \in \mathcal{F}\}$ is a family of nonempty pairwise disjoint sets (if $B_1, B_2 \in \mathcal{F}$, $B_1 \cap B_2 = \emptyset$, then $p_J(B_1) \cap p_J(B_2) = \emptyset$, where $J = J(B_1) \cap J(B_2)$, and $p_J^{-1}(p_J(B_i)) \supseteq A(B_i)$, $i=1,2$). But for $\mathcal{A}$ it is known that it satisfies the continuum chain condition

[1, Theorem 3.13]. This ends the proof.

We say that a measure μ defined on $\mathcal{B}(I)$ is given by a point if there is $x \in X(I)$ such that $\mu(B) = 1$ in case $x \in B$ and $\mu(B) = 0$ otherwise.

Let x_0 denote a point each coordinate of which is zero.

The answer to the above mentioned Frolík-Pachl conjecture is given in the following

Proposition. If $\operatorname{card}(I) > 2^c$, where c stands for the continuum, then the measure μ defined on $\mathcal{B}(I)$ by the point x_0 is not purely $\aleph_0$-compact.

Proof. Assume, a contrario, that μ is purely $\aleph_0$-compact. There is an $\aleph_0$-compact algebra $\mathcal{R} \subseteq \mathcal{B}(I)$ satisfying

$$(1) \qquad \mu(B) = \inf \{ \sum_{i=1}^{\infty} \mu(R_i) : \bigcup_{i=1}^{\infty} R_i \supseteq B,\ R_i \in \mathcal{R} \} \text{ for } B \in \mathcal{B}(I).$$

Put

$$\mathcal{R}_0 = \{ R \in \mathcal{R} - \{\emptyset\} : (R_1 \subseteq R,\ R_1 \in \mathcal{R} \text{ imply } R = R_1 \text{ or } R_1 = \emptyset) \}.$$

$\mathcal{R}_0$ contains pairwise disjoint nonempty sets, hence by (ii) is of cardinality at most c.

Claim. For any $R \in \mathcal{R} - \{\emptyset\}$ there is $R_0 \in \mathcal{R}_0$, $R_0 \subseteq R$. Suppose not. There is a set $R \in \mathcal{R}$ such that R and all its nonempty subsets belonging to $\mathcal{R}$ can be divided into two nonempty sets contained in $\mathcal{R}$. Let $R(0), R(1) \in \mathcal{R} - \{\emptyset\}$ be two disjoint sets such that $R = R(0) \cup R(1)$. If we have a family $\{R(e_1,\dots,e_i) : e_1,\dots,e_i \in \{0,1\},\ i=1,\dots,N\} \subseteq \mathcal{R}$ satisfying

$$(2) \begin{cases} R(e_1,\dots,e_i,0) \cap R(e_1,\dots,e_i,1) = \emptyset \\ R(e_1,\dots,e_i,0) \cup E(e_1,\dots,e_i,1) = R(e_1,\dots,e_i) \end{cases}$$

for $i < N$, then in each set $R(e_1,\dots,e_N)$ we can find two its subsets $R(e_1,\dots,e_N,0)$, $R(e_1,\dots,e_N,1) \in \mathcal{R} - \{\emptyset\}$ disjoint and with sum equal to $R(e_1,\dots,e_N)$.

Let $\mathcal{C}$ be the σ-algebra generated by a family $\{R(e_1,\dots e_i): e_1,\dots e_i \in \{0,1\},\ i = 1,2,\dots\} \subseteq \mathcal{R} - \{\emptyset\}$ satisfying (2). $\mathcal{C}$ is obviously countably generated. Any sequence $e_1, e_2,\dots$ where $e_i \in \{0,1\}$, defines an atom of $\mathcal{C}$ - namely $\bigcap_{i=1}^{\infty} R(e_1,\dots,e_i)$ - nonempty because of compactness of $\mathcal{R}$. So $\mathcal{C}$ has uncountably many atoms which contradicts (i). This contradiction proves the claim.

With each set $R \in \mathcal{R}$ we can associate a family $\{R_i \in \mathcal{R}_0 : R_0 \subseteq R\}$. By the claim different sets have different families, then there are at most 2^c many sets in $\mathcal{R}$. While for any set B in $\mathcal{B}(I)$ the set $J(B)$ is countable, the set $J = \bigcup\{J(R): R \in \mathcal{R}\}$ has cardinality at most 2^c. For any $i \in I$ $\mu(B(\ )) = 0$, where $B(i)$ is the set of points whose i-th coordinate is equal to 1. By (1) there is a countable family $\mathcal{R}_i \subseteq \mathcal{R}$ which covers $B(i)$ and does not cover the point x_0. There is a set $R_i \in \mathcal{R}_i$ containing a point x_i, the point which differs from x_0 only on the i-th coordinate. Hence i must belong to $J(R_i)$, and then $I = J$. This implies $\mathrm{card}(I) \leq 2^c$. This contradiction with assumption of proposition ends the proof.

Remarks. A little modification is needed to show that the proposition is true for any measure on $\mathcal{B}(I)$ defined by a point. It may be shown that any 0-1 measure on $\mathcal{B}(I)$ is defined by a point. Property (i) implies that any measure on $\mathcal{B}(I)$ is at most countable sum of two-valued measures, so everyone is pure ([2, Lemma 2.2]) and hence $\aleph_0$-compact

([3, Corollary 4]) but none is purely $\aleph_0$-compact.

References

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